Complex Analysis: Exercise 7

1. Let $Q = \{x + iy \in \mathbb{C} : 0 < x, y < 1\}$ be the interior of the unit square, and let $\gamma : [0, 1] \to \mathbb{C}$ be a continuously differentiable path in $\mathbb{C}$.

   - Show that there exists a point $a \in Q$ which is not on γ. That is, $a \neq \gamma(t)$ for all $t \in [0, 1]$.
   - Assume that both $\gamma(0)$ and $\gamma(1)$ are not in Q. Show that there exists a homotopy which is constant for all points of γ not in Q, but which moves γ to a path which does not meet Q.

2. Let $w \in \mathbb{C}$ be an arbitrary non-zero complex number. Show that there are infinitely many different numbers $z \in \mathbb{C}$ such that

   $$e^{\frac{1}{z}} = w.$$

   In particular, for all $\epsilon > 0$ there are infinitely many such z with $|z| < \epsilon$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. That is, f is continuously differentiable at all points $z \in \mathbb{C}$. For $z \neq 0$ we define the function

   $$g(z) = f\left(\frac{1}{z}\right).$$

   Under what conditions is 0 a pole or a removable singularity of the function g?