A Class of Partially Ordered Sets: III

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1 Definitions

To begin with, let us fix a certain notation. Let the pair $(X, \leq)$ be a given poset. For $a \in X$, let $a_\downarrow = \{x \in X : x < a\}$ and $a_\uparrow = \{x \in X : x > a\}$. Furthermore, let $a_\Downarrow = \{x \in X : x \leq a\}$ and $a_\Uparrow = \{x \in X : x \geq a\}$. More generally, if $Y \subset X$ is given, then $Y_\downarrow = \{x \in X : x < y, \forall y \in Y\}$. Also $Y_\Downarrow = \{x \in X : \exists y \in Y \text{ with } x \leq y\}$. The sets $Y_\uparrow$ and $Y_\Uparrow$ are defined similarly.

Given two elements a, $b \in X$, we write $a \| b$ to mean that both $a \not\leq b$ and $b \not\leq a$. Also $a \perp b$ means that either $a \leq b$ or else $b \leq a$.

An element $a \in X$ is a *minimal* element if $a_\downarrow = \emptyset$. Similarly, a is *maximal* if $a_\uparrow = \emptyset$. Minimal and maximal elements are *extremal* elements. Elements which are not extreme are *interior* elements. If a non-empty poset has no extreme elements, then obviously it must be infinite.

The poset is called *connected* if for any two elements a, $b \in X$, there exists a (finite) sequence of elements of X, starting with a and ending with b, such that adjacent pairs of elements in the sequence are always related.

The usual definition of "discreteness" is that the set $a_\downarrow \cap b_\uparrow$ should be finite, for all possible $a, b \in X$. However this is *not* the definition we will use. Instead we will say that X is *discrete* (or *strongly discrete* if we would like to emphasize the difference with the usual definition) if $a_\downarrow \setminus b_\downarrow$ (the set difference) is finite, for all a, $b \in X$.

Furthermore, we will be interested in a condition which is analogous to the axiom of *extensionality* in set theory. A poset X will be called extensional if for all a, $b \in X$ with $a \neq b$ we have $a_\downarrow \neq b_\downarrow$.

The poset X will be called *confluent below* if for any two elements a, $b \in X$ we have $a_\downarrow \cap b_\downarrow \neq \emptyset$.

The poset X will be called *upwardly seperating* if for any three elements a, b and $c \in X$ with both $a \not> c$ and $b \not> c$, we have $(a_\uparrow \cap b_\uparrow) \setminus c_\uparrow \neq \emptyset$. In particular, this implies the simpler condition that $a_\uparrow \setminus b_\uparrow \neq \emptyset$, for all $a \| b$. (Note that the condition that a poset is upwardly seperating implies by itself that the poset contains no maximal elements.)

Definition 1. *Let us denote by $\mathfrak{W}$ the class of all discrete, upwardly seperating, confluent below, connected posets X, fulfilling the condition of extensionality, such that all elements of X are interior.*

So one sees that the theory of posets in the class $\mathfrak{W}$ is really nothing other than the theory of finite sets, with the difference that the empty set is excluded. That is, we are dealing with set theory, but *without* Zermelo-Fraenkel's axiom of regularity.

From now on, when taking some poset X, we will generally assume that X belongs to the class $\mathfrak{W}$, at least unless otherwise stated or implied by the context.

Within the theory of posets, the ideas of chains and antichains are important. A *chain* $C \subset X$ is simply a totally ordered subset. (Of course the subset C here inherits the ordering relations of the containing poset.) An antichain is a subset $A \subset X$ such that for any two elements a, $b \in A$ with $a \neq b$, we have $a \| b$.

A chain is *maximal* if it cannot be properly contained in another chain. Similarly a maximal antichain cannot be properly contained in another antichain. Obviously, a maximal antichain $Y \subset X$ is such that $X = Y_\Downarrow \cup Y_\Uparrow$.

If we assume that the poset X (in $\mathfrak{W}$) contains a finite maximal antichain Y then, since X is discrete, all antichains contained within $Y_\Downarrow$ are also finite. Furthermore, given any $x \in Y_\Uparrow$, where Y is a finite maximal antichain, then there exists a finite maximal antichain containing x. To see this one only need observe that the set $x_\downarrow \cap Y_\Uparrow$ must be finite. Then take the set of least elements above $x_\downarrow \cup Y$. (That is, the minimal elements in the subset of X consisting of $X \setminus (x_\downarrow \cup Y_\Downarrow)$.) The fact that X is confluent below and extensional implies that this set must be finite.

Since X is extensional, we cannot have a maximal antichain consisting of just a single element. But much more than this, we have the following theorem.

Theorem 1. *For every poset in $\mathfrak{W}$, all maximal chains and maximal antichains are infinite.*

Proof. The fact that maximal chains are infinite follows trivially from the condition that all elements are interior elements. To see that maximal antichains are also infinite, assume to the contrary that $A \subset X$ is a finite maximal antichain. Since X is confluent below, there exists an element a which is less than all elements of A. Choose an element $b \in X$ with $b \| a$. Such an element b must exist since we can choose some $d < a$ with $d_\uparrow \cap a_\downarrow = \emptyset$ and then observe that the set $d_\uparrow \setminus a_\uparrow \neq \emptyset$, since X is upwardly seperating. Similarly we have $b_\uparrow \setminus a_\uparrow \neq \emptyset$. In fact though, the set $b_\uparrow \setminus a_\uparrow$ must be finite. To see this, begin by noting that $A_\Downarrow \cap b_\uparrow$ must be finite. Then observe that if $c \notin A_\Downarrow$, yet $c \in b_\uparrow \setminus a_\uparrow$, then we cannot have $c \in A_\Uparrow$ (for then we would have $c > a$), therefore $c \| A$. But this is also impossible, since A is assumed to be a maximal antichain.

Thus $b_\uparrow \setminus a_\uparrow$ must be finite but non-empty and so we can choose a maximal element $d \in b_\uparrow \setminus a_\uparrow$. But then we must have $d_\uparrow \setminus a_\uparrow = \emptyset$, which contradicts the condition that X is upwardly seperating. $\square$

2 Positions

Given any poset $(X, \leq)$ — not necessarily in $\mathfrak{W}$ — then we can define the set of positions within X as follows.

Definition 2. *A position $\mathcal{P} \subset X$ consists of a pair of non-empty subsets U, $V \subset X$, such that $U \leq V$ (that is, $u \leq v$ for all $u \in U$ and $v \in V$) and such that the pair is maximal in the sense that if U is properly contained in U', then we cannot have $U' \leq V$, and also if V is properly contained in V' then we cannot have $U \leq V'$. We also write $\mathcal{P}_\downarrow$ for U and $\mathcal{P}_\uparrow$ for V.*

Obviously, given any element $a \in X$, then the pair $(a_\Downarrow, a_\Uparrow)$ forms a position in X. We will call such positions *elementary positions*. The set $\Omega(X)$ of all possible positions in X is itself a poset in a natural way. It contains X, but in general it is much larger than X. One could say that $\Omega(X)$ is the *completion* of X. Within the theory of finite posets, if we add in a single minimal element and a single maximal element, then the completion is the *Macneille completion*, which is a lattice.

Given the position $\mathcal{P} = (U, V)$, then it is obviously determined by its lower and upper sets U and V. After all that is the *definition* of the position. But it may be possible to find two subsets $U_* \subset U$ and $V_* \subset V$ such that $\mathcal{P}$ is the *only* position lying between U_* and V_*. In this case we can say that $\mathcal{P}$ is determined by the pair (U_*, V_*).

Definition 3. *Let U_*, $V_* \subset X$ be two subsets, such that $U_* \leq V_*$. If there is only one position $\mathcal{P}$ in X such that $U_* \leq \mathcal{P} \leq V_*$, then we will say that $\mathcal{P}$ is* determined by *the pair (U_*, V_*). The pair will be called* minimal*, if there is no smaller pair (Y, Z) with $Y \subset U_*$ and $Z \subset V_*$ which also determines $\mathcal{P}$. When considering pairs of subsets which determine a given position, we will usually assume that the pair is minimal.*

Now it is obvious that each elementary position is determined by just one single element, namely the element which the position represents. Furthermore, if $\mathcal{P}$ is a non-elementary position, determined by the minimal pair (Y, Z), then both Y and Z must have at least two elements.

Another way to look at these things is the following. Let $U_* \subset X$ be some subset such that $U_{*\uparrow} \neq \emptyset$. Then take $V = U_{*\uparrow}$, and $U = V_\downarrow$. If we assume that $U_{*\uparrow}$ is not the upper set of some element of X, then the pair (U, V) is a position $\mathcal{P}$ in X. Therefore, given that $\mathcal{P}$ is determined by some pair (U_*, V_*), then we can also say that $\mathcal{P}$ is determined by the lower set U_* alone, following this procedure. Analogously, a position can be determined by an upper set.

At this stage, it is useful to consider a further idea.

Definition 4. *Let $(X, \leq)$ be a poset (again, not necessarily in $\mathfrak{W}$), and let $\mathcal{P}$ be a position in X. We will say that an element $a \in X$ is* associated with *$\mathcal{P}$ if $\mathcal{P} \setminus \{a\}$ is* not *a position in $X \setminus \{a\}$. If an elementary position is associated with itself (that is, with the element generating the position), then we will say that the element is an* essential *element. Otherwise, the element is non-essential; it can simply be removed without affecting the set of positions of X.*

We now confine our attention to posets in our class $\mathfrak{W}$.

Theorem 2. *Let $a \in X$ be associated with the non-elementary position $\mathcal{P} \subset X$. Then $a < \mathcal{P}$. That is $a \in \mathcal{P}_\downarrow$.*

Proof. Let $\mathcal{P} = U \cup V$ with $U \leq V$. If $a \notin \mathcal{P}$ then $\mathcal{P} \setminus \{a\} = \mathcal{P}$. Since a is associated with $\mathcal{P}$, it must be that the pair (U, V) is not maximal in $X \setminus \{a\}$. But that implies that (U, V) is not maximal in X, which is a contradiction.

If $X \in \mathfrak{W}$ and if $a \geq \mathcal{P}$ (that is, $a \in V$, the upper set of the position), then since $\mathcal{P} \setminus \{a\}$ is not a position in $X \setminus \{a\}$, it must be that the pair $(U, V \setminus \{a\})$ is not maximal in $X \setminus \{a\}$. That is, there must be an element $b < V \setminus \{a\}$, such that $b \notin U$, and so $b \| a$. But since X is upwardly separating, there exists some $c \in a_\uparrow \setminus b_\uparrow$. Since a is in the upper set of $\mathcal{P}$, we must have c also being in the upper set. i.e. $c \in V \setminus \{a\}$. However, this contradicts the fact that $b < V \setminus \{a\}$. $\square$

Theorem 3. *Let a, $b \in X$, and $\mathcal{P}$ be a position in X such that $a \not> b$ and $\mathcal{P} \not> b$. If $X \in \mathfrak{W}$ then $(a_\uparrow \cap \mathcal{P}_\uparrow) \setminus b_\uparrow \neq \emptyset$.*

Proof. If $\mathcal{P}$ is an elementary position, then this is just the definition of upwardly seperating. Assume therefore that $\mathcal{P}$ is non-elementary. If $a \perp \mathcal{P}$ then either $a_\uparrow \cap \mathcal{P}_\uparrow = a_\uparrow$, or else $a_\uparrow \cap \mathcal{P}_\uparrow = \mathcal{P}_\uparrow$. In either case, since X is upwardly seperating, we must have $(a_\uparrow \cap \mathcal{P}_\uparrow) \setminus b_\uparrow \neq \emptyset$.

If $a \| \mathcal{P}$ then since $\mathcal{P} \not> b$, there must exist an $x > \mathcal{P}$ with $x \| b$. Then $\emptyset \neq (a_\uparrow \cap x_\uparrow) \setminus b_\uparrow \subset (a_\uparrow \cap \mathcal{P}_\uparrow) \setminus b_\uparrow$. $\square$

Theorem 4. *Assume that the non-elementary position $\mathcal{P}$ is determined by the minimal pair (U_*, V_*), where $U_* < V_*$. Assume furthermore that A is the set of all elements of X which are associated with $\mathcal{P}$ (and therefore $A \subset \mathcal{P}$). Then we have $A \subset U_*$.*

Proof. Let $a \in A$. Since $\mathcal{P}$ is associated with a, we must have another position $\mathcal{R}$ in X with $\mathcal{P}_\downarrow = \mathcal{R}_\downarrow \cup \{a\}$. If $a \notin U_*$ then we would have both $\mathcal{P}$ and also $\mathcal{R}$ being between U_* and V_* so that $\mathcal{P}$ is not determined by the pair (U_*, V_*). This is a contradiction. $\square$

Conversely, we have

Theorem 5. *With the same assumptions as before, $U_* \subset A$.*

Proof. Let $u \in U_*$. The problem is to show that $u \in A$. Since the pair (U_*, V_*) is minimal, if we remove u from U_* then there must exist some other position $\mathcal{Q}$ with $\mathcal{Q} \neq \mathcal{P}$, such that $U_* \setminus \{u\} \subset \mathcal{Q}_\downarrow$ and $V_* \subset \mathcal{Q}_\uparrow$. So we choose $\mathcal{Q}$ to be the greatest position which is less than $\mathcal{P}$, yet $U_* \setminus \{u\} \subset \mathcal{Q}_\downarrow$.

Now, if $u \in A$ then we are finished. Otherwise, $\mathcal{P} \setminus \{u\}$ remains a position in $X \setminus \{u\}$. In this case, $\mathcal{P}_\downarrow \setminus \mathcal{Q}_\downarrow$ must contain more elements than just u. So let $v \in \mathcal{P}_\downarrow \setminus \mathcal{Q}_\downarrow$, with $v \neq u$. We must either have $v \not> u$ or else $u \not> v$. If $v \not> u$ then since X is upwardly separating, there exists some $b \in (v_\uparrow \cap \mathcal{Q}_\Uparrow) \setminus u_\uparrow$. In particular, $b \not> \mathcal{P}$. Then let $\mathcal{Q}'$ be a position less than $\mathcal{P}_\uparrow \cup \{b\}$ and greater than $\mathcal{Q}_\downarrow \cup \{v\}$. We must have $\mathcal{P} > \mathcal{Q}' > \mathcal{Q}$, which is impossible, since $\mathcal{Q}$ was chosen to be a maximal position beneath $\mathcal{P}$. The case that $u \not> v$ is analogous. □

Therefore we see that, for posets in $\mathfrak{W}$, the elements associated with a position uniquely determine that position. Furthermore they always lie beneath the position. Given the position $\mathcal{P}$, and an element $a < \mathcal{P}$ with which it is associated, then $\mathcal{P}_\downarrow \setminus \{a\}$ is the lower set of a new position $\mathcal{Q}$ in X, lying directly beneath $\mathcal{P}$. Of course, $\mathcal{Q}$ may be associated with quite different elements than those which were associated with $\mathcal{P}$. Nevertheless, we see that it is possible to descend systematically through the positions in a poset in $\mathfrak{W}$ by successively discarding single elements which are associated with the positions.

Theorem 6. *Let $\mathcal{P}$ be a non-elementary position in the poset $X \in \mathfrak{W}$, such that $\mathcal{P}$ is associated with the elements $a_1, \dots, a_n$. Then given any finite subset $K \subset X$, there exists a finite subset $X_f \subset X$, with $K \subset X_f$, containing all the elements $a_1, \dots, a_n$, such that $\mathcal{P}_f = \mathcal{P} \cap X_f$ is a position in X_f, and $\mathcal{P}_f$ is also associated with precisely the elements $a_1, \dots, a_n$.*

Proof. For each a_i there must exist an element c_i of X with $c_i \not> a_i$, yet $c_i > a_j$, for all $j \neq i$. Choose $X_f \supset K$ so that it contains at least one such element, for each i. Then for each element $b \in X_f \setminus \mathcal{P}$, there must exist some $d > \mathcal{P}$ with $d \not> b$. Include at least one such d in X_f, for each such b. □

3 Dimension

For finite posets, the standard definition of dimension is as follows. Let $(P, \leq)$ be a poset, that is a finite set P, and an ordering relation, denoted '$\leq$', with $\leq \subset P \times P$. Different partial orderings can be assigned to P. The set of all these partial orderings is itself a partially ordered set. Any given partial ordering is contained within a maximal partial ordering, which is a total ordering of P. Such a total ordering is a linear extension of the original partial ordering. If two elements $a, b \in P$ were related in the original partial ordering, say $a \leq b$, then obviously they will still have the same relation in any linear extension. On the other hand, if $a \| b$ in the original partial ordering, then either $a < b$ or $a > b$ in any given linear extension. A *realizer* of the partial order is a set of linear extensions, denoted $\{L_1, \ldots, L_n\}$, such that for any unrelated pair $a \| b$, we have $a < b$ in one of these L_i and $a > b$ in some other L_j. The minimum number of linear extensions necessary to create a realizer is then defined to be the *dimension* of the poset $(P, \leq)$.

This definition is more appropriate for finite posets, rather than infinite posets. Let us therefore say that an infinite poset is *locally n-dimensional* if any finite subset, with the inherited partial ordering, is at most n-dimensional, and this n is the minimum such number. But then, since no other definition will be used, we can simply leave out the qualifier 'locally', and say that the poset is n-dimensional.

To begin with, it is easy to show that all posets in $\mathfrak{W}$ must be at least 3-dimensional.

Theorem 7. *Let $X \in \mathfrak{W}$. Then the dimension of X is at least three.*

Proof. Since every maximal anti-chain in X is infinite, we certainly do not have the dimension being only one. But also we can find three elements x_1, x_2, and x_3 in X which are pairwise mutually unrelated.

Since $X \in \mathfrak{W}$, for each permutation (i, j, k) of the three numbers 1, 2, 3, there exists some $y_{(i,j,k)} \in X$ with $y_{(i,j,k)}$ greater than x_i and x_j, but not greater than x_k. Choose some finite subset X_f of X containing these elements.

If X_f were two dimensional, then there would be a realizer consisting of two linear extensions, L_1 and L_2. Let us say that in the linear extension L_1 we have $x_1 < x_2 < x_3$. Then in the linear extension L_2 we must have x_3 being less than both x_1 and x_2, and furthermore, x_2 must be less than x_1. That is, we have $x_3 < x_2 < x_1$ in L_2. Since $y_{(1,3,2)} > x_3$, it must be greater than x_2 in L_1. Also, since $y_{(1,3,2)} > x_1$, it must be greater than x_2 in L_2. Therefore $y_{(1,3,2)} > x_2$ in both L_1 and L_2, hence also in X_f, which is a contradiction. $\square$

Theorem 8. *Assume that the dimension of the poset $(X, \leq)$ in $\mathfrak{W}$ is n. Then each position in X can be associated with at most n elements.*

Proof. Let $\mathcal{P}$ be some position in X, and assume that it is associated with $n+1$ different elements $\{x_1, \ldots, x_{n+1}\}$ of X. Then there is some finite subset $X_f \subset X$, having the properties that $\{x_1, \ldots, x_{n+1}\} \subset X_f$, and furthermore, there exists a position $\mathcal{R}$ in X_f which is associated with $\{x_1, \ldots, x_{n+1}\}$, such that each x_i is contained in $\mathcal{R}_{\downarrow}$. Let $\{L_1, \ldots, L_n\}$ be a realizer for X_f. For each $i = 1, \ldots, n$, let $j(i)$ be the number such that $x_{j(i)}$ is the highest element in the linear order L_i. Since there are $n+1$ elements in $\{x_1, \ldots, x_{n+1}\}$, it must be that at least one of them, say x_{n+1}, is not the highest element in any of the linear orders in the realizer. But then, given any element $p \in X_f$, with $p > \{x_1, \ldots, x_n\}$, we must have $p > x_{n+1}$, since it has this relation throughout the realizer. Therefore we must have $x_{n+1} < \mathcal{R}$, for all positions $\mathcal{R}$ which contain $\{x_1, \ldots, x_n\}$, and so if $\mathcal{R}$ is associated with x_j, for $j = 1, \ldots, n$, then it cannot be associated with x_{n+1}. $\square$

Theorem 9. *Again assume that $(X, \leq)$ is a poset in $\mathfrak{W}$, which has dimension n. We add in a further element p to X to obtain the larger set $X' = X \cup \{p\}$. The partial ordering of X is also extended by including new ordering relations involving the element p in such a way as to make X' a poset in $\mathfrak{W}$. Assume furthermore that in this extended poset X', we have that if $q \in X$ with $q \| p$ then both the sets $q_\downarrow \backslash p_\downarrow$ and $p_\downarrow \backslash q_\downarrow$ are not empty. Then X' also has dimension n.*

Proof. Assume to the contrary that there exists some finite subset $K \subset X'$ with dimension greater than n. Then we must have $p \in K$. For each $q \in K$ with $q \| p$, there must be some $u \in X$ with $u \in p_\downarrow \setminus q_\downarrow$. Include one such u into K for each such q. Since X is upwardly seperating and $u \| q$, there must be some $v \in X$ with $v \in q_\uparrow \setminus u_\uparrow$. So include such a v into K as well. Therefore we have $u \| v$ and also $u < p$ and $v > q$.

Similarly we can find u', v' in X such that $u' \| v'$ with $u' < q$ and $v' > p$. Add these elements in to K as well.

Now $K \setminus \{p\}$ has dimension at most n. Therefore, let $\{L_1, \ldots, L_n\}$ be a realizer for $K \setminus \{p\}$. The extra element p can be included into each of the linear extensions L_i in some way. But note that since $u \| v$, we must have $u > v$ in one of the linear extensions, say in L_j. Therefore in L_j we have $p > u > v > q$. On the other hand, by symmetry, one of the other linear extensions, say $L_{j'}$, has $q > u' > v' > p$. Therefore our set of n linear extensions $\{L_1, \ldots, L_n\}$, when restricted to the original subset K, is a realizer for that K. Thus the dimension of the original subset was at most n. A contradiction. $\square$

In particular, an obvious procedure would be to add in the new element p to X in such a way that $p_\downarrow$ is the lower set of a non-elementary position in X.

Thinking about this, it is interesting to consider constructing finite subsets of posets in $\mathfrak{W}$ in some systematic way. For example, one might begin with some finite set $M = \{x_1, \ldots, x_n\}$, of unrelated elements of X. Then, starting from M as a set of lowest elements, we might add in more elements of X above M, doing so in such a way that these elements have lower sets which are non-elementary positions in the finite set, as it has been constructed up to that stage. The possibilities then depend upon the number n of elements in the starting set M.

- If M only consists of a *single* element x_1, then the construction is very limited. The only possibility is to construct the element $a > x_1$ directly above x_1, so that there is no element between them. However, in this case we have $a_\downarrow = \{x_1\}$, which is the lower set of x_1 itself. Therefore no construction is possible.

- If M consists of two elements $M = \{x_1, x_2\}$, then if we attempt to construct a new element a above M, the only choice is to have $a_\downarrow = M$, and so the construction again comes to a stop.

- If M has three elements, $\{x_1, x_2, x_2\}$, then we either construct the first element a above all three, but this again leads to a stop in the construction. Alternatvely, a can be taken to be above two of the elements of M, but not above the third. This gives three possibilities, but once they have been constructed we can only construct three on top of them, and so forth. This leads to a simple "generalized" chain.

- So it is only when M has at least four elements that a non-trivial poset can be constructed, at least under these assumptions. With some additional assumptions, this minimal number of 4, if taken, leads to the constructed poset having dimension 4.

4 Variations

Let $(X, \leq)$ be a poset in $\mathfrak{W}$, and let $a \| b$ be two unrelated elements in X such that $a_\downarrow \setminus b_\downarrow = \emptyset$. For any given b, there must exist such a corresponding a, owing to the fact that X is discrete.

Next, take $\mathcal{P}$ to be the greatest position which is less than both a and b. That is, let $U = \{u \in X : u \in a_\downarrow \cap b_\downarrow\}$, and then let $V = \{v \in X : v > U\}$. The pair (U, V) is the position $\mathcal{P}$.

If $\mathcal{P}$ is a non-elementary position, then one possible variation which we might consider is obtained by removing the element a from X, and then inserting it into the position $\mathcal{P}$. Let us denote the new poset thus obtained X'. Since in X we had $a > \mathcal{P}$, it follows that a was not associated with $\mathcal{P}$ in X so that it remains a position in X'.

It is obvious that X' is a discrete poset. However, it is possible that the property of extensionality is lost in X'. This would be the case if there are two elements $p \| q$ in X with $p_\downarrow = q_\downarrow \cup \{a\}$. In this case we might perform a further variation so that the relation $q < p$ results. It might also be the case that X' is not upwardly seperating. This would be true if there were two elements u and v, with $u \not> a$ and $v \not> a$, and $(u_\uparrow \cap v_\uparrow) \setminus a_\uparrow \subset b_\uparrow$.

But generally speaking, we will say that any finite reordering of the elements of X, producing a new poset in $\mathfrak{W}$, will be called a *valid variation.* (Finite here in the sense that if we examine the pairs of elements of X which experience a change in their ordering relations, then there is a finite subset of X such that each of the pairs has at least one member in that finite subset.)

5 Positions vs. Elements

Up till now, we have not specified any conditions which would tend to favor one poset in $\mathfrak{W}$ over another. But for various reasons, it seems interesting to investigate the class of posets in $\mathfrak{W}$ which are such that the proportion of positions to elements is as low as possible. Since all posets in $\mathfrak{W}$ are infinite, it follows that there are always infinitely many positions and elements. Therefore, in order to compare the proportion of positions to elements, we need a method which only involves counting finite subsets.

Definition 5. *Let $X \in \mathfrak{W}$. Given a, $b \in X$, then the subset $B(a,b) = a_\uparrow \cap b_\downarrow$ is finite. Now let X' be a variation of X in the sense that X' is identical with X in $X \setminus B(a,b)$. But in $a_\uparrow \cap b_\downarrow$, X' might differ from X. In fact in X', we have that $a_\uparrow \cap b_\downarrow = B'$, where B' is in general different from $B(a,b)$. On the other hand, we require B' to have the same number of elements as $B(a,b)$, and furthermore, we require that $X' \in \mathfrak{W}$. Given this, then we say that X' is an* admissible variation *of X between a and b.*

So given two elements $a < b$ in X, it is easy to compare the proportion of positions to elements between a and b. There can only be finitely many of each, so we only need to count them. The poset X will have the smallest proportion of positions to elements — at least between a and b — if there is no admissible variation between a and b, such that the number of positions between a and b in the varied poset X' is less than the corresponding number in X.

But there is a problem. It may be that X is optimal between the two points $a < b$ in the sense that the number of positions there is as small as possible with respect to admissible variations between a and b. But then we can always take two further elements $a' < b'$, such that $a' < a < b < b'$, so that admissible variations between a' and b' include those between a and b. And then it might be that for such a pair, a' and b', there does exist a variation X', having fewer positions between a' and b'. This varied poset could be different between a to b, even though in this varied version, the number of positions between a and b might be greater. In fact, it may even be the case that no elements in the varied poset X' correspond to our original pair a and b.

Since X is upwardly separating, given elements like a and b, they can be located in a larger variation X' by examining their upper sets $a_\uparrow$ and $b_\uparrow$ outside the varied region, assuming that corresponding elements a' and b' exist in X'. This leads to the following definition.

Definition 6. *The poset $X \in \mathfrak{W}$ will be called* dense *if for any two related elements $a < b$ there exist further elements $a' < a < b < b'$ such that for all variations of X which include a' and b', a variation with the fewest positions leaves the subset $a_\uparrow \cap b_\downarrow$ unchanged.*

Therefore, the question is, what properties do the dense posets have (assuming that dense posets exist in the first place)?

To begin with, it is always possible to consider variations of the form dealt with in the previous section. Let p, $q \in X$, with $p \| q$ and $p_\downarrow \subset q_\downarrow$. Then, in the varied version X', the relation $p \| q$ is changed to $p < q$, perhaps together with sufficient further relations involving p to ensure that X' is still in $\mathfrak{W}$.

Does X' have fewer positions than X? Since the only difference between X' and X is the fact that in X', there are more relations involving the element p, it is obvious that we need only consider positions with are associated with p. But, as we have seen, positions are determined by their lower sets. And the variation from X to X' only involves changes *above* the element p. Therefore we need only examine positions above p, which are associated with p. (It may be that in the variation, taking X to X' still more new ordering relations, not involving p, are necessary. But they must also be above p in X. Such additional orderings will not be considered here.)

For simplicity, let us consider only the elements p and q in relation to possible positions in X and X'.

To begin, take two positions, $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{B} < \mathcal{A}$. Then we can descend from $\mathcal{A}$ to $\mathcal{B}$ by observing first that $\mathcal{A}$ is associated with various elements $a_1, \ldots, a_n$. So if we remove one of these elements, say p_1 from $\mathcal{A}_\downarrow$, where $a_1 \not< \mathcal{B}$, then we descend to a lower set $\mathcal{A}'$ say, which has $\{a_2, \ldots, a_n\} \subset \mathcal{A}'$ and $\mathcal{A}' \geq \mathcal{Q}$. The procedure is then repeated, using $\mathcal{A}'$, rather than $\mathcal{A}$. In this way we descend through finitely many steps from $\mathcal{A}$ to $\mathcal{B}$, at each stage simply removing one single element with which the position is associated.

In particular, if we start from a position $\mathcal{A}$ above our two elements p and q, then descend to a position $\mathcal{B}$ below both of the elements, then we can compare the steps in X, and in X'. There are generally many ways to do this. However, let us consider what happens if, at some stage, descending through X, we reach a position $\mathcal{P}$ which is associated with *both* p and q?

In this case, we can think about three different positions which will occur in X, namely $\mathcal{P}$ itself, then the position $\mathcal{R}$, whose lower set is $\mathcal{P}_\downarrow \setminus \{q\}$, and finally the position $\mathcal{Q}$, whose lower set is $\mathcal{P}_\downarrow \setminus \{p\}$. On the other hand, in X', the position $\mathcal{P}$ will have disappeared; it will have merged into $\mathcal{Q}$. Furthermore, $\mathcal{R}$ may, or may not, have disappeared.

Thus it seems reasonable to consider the following property. Namely, we expect that a dense poset X does not contain any pairs of elements p and q, with $p \| q$ and $p_\downarrow \setminus q_\downarrow = \emptyset$, such that there exists a position $\mathcal{P}$ in X which is associated with both p and q and such that the variation which is given by indroducing the extra relations implied by $p < q_\Uparrow$, produces a poset in $\mathfrak{W}$.

Definition 7. *Let us call a poset $X \in \mathfrak{W}$ having this property a* conditionally dense poset*.*

The question of whether or not conditionally dense posets are necessarily dense, or conversely, remains perhaps unclear. Nevertheless, this definition gives us a practical method for testing finite configurations which might be subsets of dense posets. So the question is, what properties do conditionally dense posets have?

6 Finite models for conditionally dense posets

The problem with our theory is that the posets in $\mathfrak{W}$ are always infinite. Yet we would like to know which finite configurations might occur in possible dense, or at least conditionally dense posets in $\mathfrak{W}$. Computer experiments, calculating possible variations, cannot be performed with infinite posets since computers only have finite memories. Therefore it may be useful to change the framework of our theory to allow finite posets, and thus enable us to think about computer experiments with finite models.

The experiment we have in mind is the following. Begin by taking some random finite poset X, which the computer generates using some arbitrary procedure. The computer should then check that X is connected. Furthermore, we require that for all pairs of interior elements x and y of X, we have $x_\downarrow \neq y_\downarrow$. This gives us a kind of extensionality property for the finite model X. Then we must check that X is future separating, in the sense that for all interior triples of elements x, y and z with $x \not> z$ and $y \not> z$, we always have $(x_\uparrow \cap y_\uparrow) \setminus z_\uparrow \neq \emptyset$. Given that we have found such a finite model, then obviously we would expect it to have a great number of maximal elements in comparison with the number of its interior elements.

Assuming that we have found such a finite poset X, then we would like to see if it is possible to simplify X using some system of admissible variations which give us a finite analog of a conditionally dense poset in $\mathfrak{W}$. To begin with, it is reasonable to require that such an admissible variation would only be allowed with respect to the interior elements of X. Therefore, given that we have two interior elements p and q in X with $p_\downarrow \setminus q_\downarrow = \emptyset$, such that there are no positions of X which are associated with both p and q, and such that the variation which introduces all the relations implied by $p < q_\Downarrow$ produces an extensional poset (with respect to the interior elements), then we will consider this to be an admissible variation.

Note that the property of being upwardly separating (for the interior elements of X) may become lost when such an admissible variation is performed. We consider two ways different of dealing with such a possibility in a computer simulation.

- On the one hand, we could say that if X is no longer upwardly separating after the variation, then this variation should not be taken after all; we will no longer consider it to be an admissible variation. Unfortunately though - as a practical matter - at each stage of the calculation, the computer would then be forced to check through many triples of interior elements to see if any instances of a violation of the condition of upward separation occurs in the varied poset. This would involve an extreme increase in the complexity of the calculation.

- The alternative is to simply ignore the question of whether or not the interior elements of the varied poset satisfy the condition of upward separation. This procedure might be justified by observing that if the finite model X were to actually occur as a finite subset of some poset $W \in \mathfrak{W}$, then there will always be infinitely many elements in the upper sets $x_{\uparrow}$, for all $x \in X$, when considered in W. The reason that the variation of the finite model X might not be upwardly separating is that there might be two interior elements a and b with $a \not> p$ and $b \not> p$, yet $(a_{\uparrow} \cap b_{\uparrow}) \setminus p_{\uparrow} \subset q_{\uparrow}$ in X. Yet that does not rule out the possibility that in the infinite set W, we do actually have $(a_{\uparrow} \cap b_{\uparrow}) \setminus p_{\uparrow} \not\subset q_{\uparrow}$, owing to the possible existence of many extra elements of W, upwards of X.

  Using this procedure, a computer program could rapidly find all variations needed to reduce a randomly given poset satisfying our initial conditions to one which is conditionally dense.