A Class of Partially Ordered Sets: V
(Results concerning probabilities)

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1 Introduction

The purpose of this paper is to make more precise some of the assertions made in previous papers. We begin with a short summary of the basic framework which is to be used, and the properties of the partially ordered sets which will be particularly needed.

Let $(X, \leq)$ be a partially ordered set (poset). We use the following notation: given $x \in X$, then $x_\downarrow = \{y \in X : y < x\}$, $x_\Downarrow = \{y \in X : y \leq x\}$, $x_\uparrow = \{y \in X : y > x\}$, and $x_\Uparrow = \{y \in X : y \geq x\}$. Furthermore, if $U \subset X$, then $U_\downarrow = \{y \in X : y < u, \forall u \in U\}$, and $U_\Downarrow = \{y \in X : y \leq u, \forall u \in U\}$. The notation $U_\uparrow$ and $U_\Uparrow$ is analogous. Given a pair of elements x and y in X, then the notation $x \| y$ means that both $x \not\leq y$ and $x \not\geq y$. An element x will be called a minimal element if $x_\downarrow = \emptyset$. It is a maximal element if $x_\uparrow = \emptyset$. If an element is either minimal or maximal, then we will say that it is *extreme*.

The poset $(X, \leq)$ will be called *discrete* if $x_\downarrow \setminus y_\downarrow$ is finite, for all pairs of elements x and y in X. It is *upwardly separating* if for all x and y with $x \not\geq y$, we have $x_\uparrow \setminus y_\uparrow \neq \emptyset$. It is called *confluent below* if for all x and y, we have $x_\downarrow \cap y_\downarrow \neq \emptyset$. It is called *extensional* if $x_\downarrow = y_\downarrow \Rightarrow x = y$.

Therefore, let $\mathfrak{W}$ denote the class of all non-trivial, discrete, upwardly separating, confluent below, extensional, connected posets which contain no extreme elements. From now on, it will be assumed that all posets which we consider will be elements of $\mathfrak{W}$.

In previous papers it was shown that if $X \in \mathfrak{W}$, then both the height and the width of X must be infinite. That is, any maximal chain (totally ordered subset), and also any maximal anti-chain (set of mutually unrelated elements) must be infinite.

2 Positions

Given some poset $(X, \leq)$ in $\mathfrak{W}$, and two non-empty subsets U, $V \subset X$, we will say that the pair (U, V) is a *position* in X if $U \leq V$ (that is, $u \leq v$, for all elements $u \in U$ and $v \in V$) such that if (U', V') is another pair with $U' \leq V'$ and $U \subset U'$, $V \subset V'$, then we must have $U = U'$ and $V = V'$. Thus the pair (U, V) can be thought of as being a maximal double cone in X. We denote the set of all positions in the poset $(X, \leq)$ by the symbol $\mathcal{P}(X)$.

For any element $x \in X$, the pair $(x_{\Downarrow}, x_{\Uparrow})$ is obviously a position. Such a position will be called an *elementary position*. We distinguish between two different kinds of elements of X. An element $x \in X$ will be called *essential* if the pair $(x_{\downarrow}, x_{\uparrow})$ is *not* a position in the poset $(X \setminus \{x\}, \leq)$. Otherwise the element is non-essential. Therefore it is reasonable to say that the structure of X, as a poset, remains unchanged if a non-essential element is removed. Put another way, the structure of X is essentially the same as that of the *complete* poset generated by X, namely the set of all positions in X. For finite posets, the analogous idea would be the MacNeille completion.

Theorem 1. *An element x in a poset $X \in \mathfrak{W}$ is essential if and only if there is no element $y \in X$ with $y \| x$ and $x_{\downarrow} \subset y_{\downarrow}$.*

Proof. Let $U = x_{\downarrow}$ and $V = U_{\Uparrow}$. Then the pair (U, V) must be a position since, if $z < V$ with $z \not< x$, then since X is upwardly separating, there exists some $w \in x_{\uparrow} \setminus z_{\uparrow}$. But then $w \in V$, contradicting $z < V$.

If $V = x_{\Uparrow}$ then x is not essential and in this case, for all $y \in V$, that is, for all y with $x_{\downarrow} \subset y_{\downarrow}$, we have $y > x$. On the other hand, if $V \neq x_{\Uparrow}$ then x is essential and there exists some $y \in V \setminus x_{\uparrow}$, that is, we have $x_{\downarrow} \subset y_{\downarrow}$. $\square$

More generally, we can find positions in a poset X using the following procedure. Let $U^* \subset X$ be some non-empty subset such that $U^*_{\Uparrow} \neq \emptyset$. Then take $V = U^*_{\Uparrow}$ and finally $U = V_{\Downarrow}$. Clearly the pair (U, V) which is thus produced is a position in X. Alternatively one could start with some non-empty subset $V^* \subset X$ such that $V^*_{\Downarrow} \neq \emptyset$. Then for $U = V^*_{\Downarrow}$ and $V = U_{\Uparrow}$, we again have (U, V) being a position in X.

Given a position (U, V) in X, we will say that an element x is *associated with* the position if $(U \setminus \{x\}, V \setminus \{x\})$ is not a position in $(X \setminus \{x\}, \leq)$. Thus we can say that an element is essential if it is associated with itself. As was shown in previous papers, if x is associated with the position (U, V), then $x \in U$. That is, x is below (less than or equal to) the position. Since X is discrete, a position (U, V) in X is associated with at most finitely many elements of X. We will say that the poset X is *essential* if all its elements are essential.

Theorem 2. *If the position (U, V) in the poset $X \in \mathfrak{W}$ is associated with the element $x \in X$, then x is an essential element.*

Proof. Since $(U \setminus \{x\}, V)$ is not a position in $X \setminus \{x\}$, there must be an element $y > U \setminus \{x\}$ with $y \notin V$. Therefore $y \not> x$, yet $y > x_{\downarrow}$. $\square$

Given any $X \in \mathfrak{W}$, let (S, T) be some pair of subsets of X with $S < T$. Then clearly there must be a position (U, V) in X which is such that $S \subset U$ and $T \subset V$. Thus if we have two elements $x, y \in X$ with $x \| y$, then since X is upwardly separating, there is an element $a \in x_{\uparrow} \setminus y_{\uparrow}$ Let $S = \{x\}$ and $T = \{a\}$. Any position (U, V) with $S \subset U$ and $T \subset V$ is then above x and not above y. Therefore X is also upwardly separating with respect to positions.

A condition which sharpens the relationship between positions and the elements with which they are associated is the following

Theorem 3. *Let $X \in \mathfrak{W}$. Assume furthermore that for any three elements x, y, z in X such that x is not related to either y or z, there must exist a further element a which is greater than x and greater than just one of y and z. Then every non-elementary position is uniquely determined by the elements with which it is associated. Specifically, each position is the lowest position above all of the elements with which it is associated.*

Proof. Let (U, V) be a non-elementary position in X. Since (U, V) is non-elementary, there exist at least two maximal elements in U. Call them x and y. They are unrelated, and since X is discrete, there are only finitely many elements in $U \setminus x_{\Downarrow}$, one of which is y. If y is the only element in $U \setminus x_{\Downarrow}$, then the position (U, V) is associated with y. Otherwise, let z be another element in $U \setminus x_{\Downarrow}$. Let a be above x and one of the elements y or z. If a is above z then there are fewer elements in $U \setminus a_{\Downarrow}$ than there were in $U \setminus x_{\Downarrow}$. If a is above y, then reverse the roles of y and z. Proceeding in this way, we conclude that there is an element $y \in U$ and an element $a \in X$ such that $U \setminus a_{\Downarrow} = \{y\}$. Therefore we must have y being associated with (U, V). This shows that every position is associated with at least one element.

Assume now that the position (U, V) is associated with the elements $x_1, \ldots, x_n$. Let (U', V') be a lowest possible position above $x_1, \ldots, x_n$. We have $U' \subset U$. If $U' \neq U$ then we must have $U \setminus U' \neq \emptyset$ and therefore $V' \setminus V \neq \emptyset$. Let $a \in V' \setminus V$. Then there exists some $y \in U \setminus U'$ with $a \| y$. As before, we now consider the finite set $U \setminus a_{\Downarrow}$, and we find an element in it which is associated with (U, V). However all the elements $x_1, \ldots, x_n$ are contained within $a_{\Downarrow}$. This contradiction shows that $U' = U$. $\square$

In order to describe the situation, let us say that a poset satisfying the conditions of the theorem is *strongly separating*. From now on it will be assumed that the posets of $\mathfrak{W}$ are strongly separating.

Of course every poset in $\mathfrak{W}$ contains many essential elements. This follows from the fact that such posets are discrete. But more than this, we have shown that all positions, and hence the entire structure of such a poset, is determined exclusively by the essential elements. Therefore, when considering some specific $X \in \mathfrak{W}$, we loose nothing if we simply discard all non-essential elements. They may be considered to provide a sort of skeleton of the poset X, allowing us to fill in the non-essential structure as much as we please, but adding nothing essentially new in the process.

3 Defining probabilities in $\mathfrak{W}$

The procedure for defining probabilities in *finite* sets seems obvious.Given some number n, then we consider all possible finite posets with n elements. The probabilities are then fixed by saying that all of these posets are equally likely. If we are interested in some particular configuration of elements which might occur in a given poset with n elements, then the probability for such a configuration would be the number of posets having that configuration, divided by the total number of posets with n elements.

But even in finite posets, one could argue that this simple counting procedure is not the best. Given the idea of positions, we can think about whether or not the various elements of the poset are essential or not. But non-essential elements contribute nothing new to the structure of the poset. Even if we were to discard them, their positions still remain. So should we count them, or not, when calculating probabilities? The rule we will follow is to calculate probabilities with respect to *complete* posets.

In particular, for the posets in $\mathfrak{W}$ we will take the complete posets — that is, posets consisting of all the positions in a given poset $X \in \mathfrak{W}$. Of course X has both infinite width as well as infinite height. On the other hand, both X and also $\mathcal{P}(X)$, the completion of X, are discrete so that it makes sense to consider finite sub-posets when thinking about probabilities.

So the question is, what types of sub-posets should we choose? The simplest system would be to take sets of the form $a_\downarrow \setminus b_\downarrow$, for given elements $a > b$ in the poset X. Then, in particular, all positions (U, V) lying between b and a — that is, in $a_\downarrow \cap b_\uparrow$ — are completely determined by their lower positions U, and these in turn are determined by the essential elements with which they are associated, which must all be in $a_\downarrow \setminus b_\downarrow$.

In the earlier paper *Discrete Partially Ordered Sets in Physics*, we have argued that a poset X in $\mathfrak{W}$ is more probable than other possible finite variations of X if for a given number of positions between given pairs of elements $b < a$, the number of essential elements in $a_\downarrow \cap b_\uparrow$ is large. Or put another way, for a given number of essential elements within $a_\downarrow \setminus b_\downarrow$, the number of positions between b and a is small.

In general, there will be many possible configurations of essential elements to consider, leading to the idea that one or another of these configurations will be more or less probable. However in one particular case we can assert that a certain configuration of the poset X will be less probable than a slightly varied configuration of X. The situation is the following.

4 A particular configuration

Assume we have two essential elements x and y in X with $x \| y$, yet $x_\downarrow \subset y_\downarrow$. Now we change X to the new poset X' by simply adding in a number of additional ordering relations between x and the elements in $y_\Uparrow$. That is, X' consists of precisely the same elements as X, the ordering of X' contains all of the ordering relations of X, and in addition we have all the relations given by $x < y_\Uparrow$. Adding in these extra relations gives us a new ordering on the set of elements of X. Clearly what results is still a poset, although it may no longer be an element of our class $\mathfrak{W}$. So let us call this new, varied poset $(X', \leq')$, where of course the elements of X' are the same as those of X.

We would like to be able to say that this varied poset X' has fewer positions than does X. Thus, as long as it remains in our class $\mathfrak{W}$, it would be more probable than X. A first step in this direction would be to show that, at least, X' has no *more* positions than does X. That is, to show that there exists an injective mapping $\mathcal{P}(X') \to \mathcal{P}(X)$.

Theorem 4. *There exists an injective mapping $\psi : \mathcal{P}(X') \to \mathcal{P}(X)$.*

Proof. Given $(U', V') \in \mathcal{P}(X')$, let $V = U'_\Uparrow$ in the ordering of X, and then take $U = V_\Downarrow$, again in the ordering of X. We define $\psi(U', V') = (U, V)$. To show that ψ is an injection, assume that there were two different positions (U', V') and (U'', V'') in $\mathcal{P}(X')$, such that $U'_\Uparrow = U''_\Uparrow$, when considered in X. But then, since the ordering relations of all elements $a \neq x$ with further elements in the upwards direction are the same, both in X and in X', it follows that U' and U'' can only differ in that one of these sets contains the element x, while the other set doesn't contain x. So let us say that $x \notin U'$, while $U'' = U' \cup \{x\}$. Therefore there must be some element $b \in V' \setminus V''$ with $b \not> x$ in the ordering of X'. In particular, $b \not> x$ in the ordering of X.

But then also in the ordering of X we would have $b > U'$, while $b \not> U''$. Therefore in X we must have $U'_{\Uparrow} \neq U''_{\Uparrow}$, which is a contradiction. $\square$

Theorem 5. *There exists a position in X which is associated with both x and y.*

Proof. Let (U, V) be the lowest position above both x and y. That is, $V = \{x, y\}_{\Uparrow}$ and $U = V_{\Downarrow}$. If (U, V) is not associated with x, then let $z_1 \neq y$ be an element with which (U, V) is associated. Since X is assumed to be strongly separating, there exists an element $w_1 > y$ for which either

1. $w_1 > x$, $w_1 \not> z_1$ or
2. $w_1 > z_1$, $w_1 \not> x$.

But $w_1 \not> z_1$ is impossible, for then we would have $z_1 \notin U$. Thus, $w_1 > \{z_1, y\}$, while $w_1 \not> x$. However since (U, V) is not associated with x, there must then be some further element z_2 with which (U, V) is associated, such that $z_2 \not< w_1$. Again, take $w_2 > w_1$ such that w_2 is only greater than one of x or z_2. As before, we conclude that there is a further element z_3, not equal to z_1 or z_2, with which the position (U, V) is associated. And so forth, producing an infinite sequence of elements z_i, $i \in \mathbb{N}$. However every position can only be associated with finitely many elements. Therefore we conclude that (U, V) must be associated with x. Reversing the roles of x and y shows that (U, V) is also associated with y. $\square$

Therefore our variation, changing $x \| y$ to $x < y$, will eliminate this lowest position above x and y, and so the varied poset should be more probable.

On the other hand we have seen that an element $x \in X$ is essential precisely when some other element $y \in X$ exists such that $x \| y$ and $x_{\downarrow} \subset y_{\downarrow}$. Furthermore, all posets in $\mathfrak{W}$ contain infinitely many essential elements. Therefore it is clear that if we wish to remain in $\mathfrak{W}$, then we must allow many such pairs to remain unrelated to one another.

The question then is, which pairs should be varied, adding new ordering relations, allowing us to obtain a more probable poset which is still in $\mathfrak{W}$? For this, we return to the argument showing that the posets having fewer positions in relation to the essential elements are more probable. According to theorem 5, a position (U, V) in the poset $X \in \mathfrak{W}$ will disappear in the variation which adds the new relations $x < y_{\Downarrow}$ if it is associated with *both* x and y.

Now each position in X is associated with only finitely many elements of X. Given that a position (U, V) is associated with some element $z \in X$ say, then we can define the *distance* between the position (U, V) and the element

z to be the number of essential elements in the set $U \setminus z_{\Downarrow}$. Let us assume that in a typical poset X in $\mathfrak{W}$, given two elements $a > b$ in X which are a given distance apart (that is, the number of essential elements in $a_{\downarrow} \setminus b_{\downarrow}$ is some given number N), the positions between a and b are associated, on average, with some number $n_{\mathcal{P}}$ of essential elements. Therefore $n_{\mathcal{P}}$ is a function of the number N. Furthermore, the average distance between those positions and the elements with which they are associated is some number $d_{\mathcal{P}}$. Given this, then it is clear that if we have a pair of essential elements (x, y), with $x \| y$ and $x_{\downarrow} \subset y_{\downarrow}$, which is such that the set $y_{\downarrow} \setminus x_{\downarrow}$ is much larger than $d_{\mathcal{P}}$, then it is not probable that many positions are associated with both x and y. Therefore we conclude that for a pair which is such that x is close to y, in the sense that the number of elements in the set $y_{\downarrow} \setminus x_{\downarrow}$ is small, then the variation giving the new relations $x < y_{\Downarrow}$ is probable. On the other hand, if x is far away from y, then the variation does not lead to a more probable poset. Put another way, a probable poset is such that these pairs are far apart from one another; it is improbable that a pair with $x \| y$ and $x_{\downarrow} \subset y_{\downarrow}$ is such that the number of elements in $y_{\downarrow} \setminus x_{\downarrow}$ is small. And we can take this conclusion to be true independently of which elements $a > b$ are chosen for specifying some region $a_{\downarrow} \setminus b_{\downarrow}$ within the poset X.

5 Generalized chains

If we were dealing with finite posets rather than the posets in $\mathfrak{W}$, then the results of the last section would show that the variation described there would result in fewer positions. However the same cannot always be said for infinite posets. After all, if there are infinitely many positions, then removing a few of them leaves us still with infinitely many positions. Nevertheless, it seems reasonable to assume that a probable poset in $\mathfrak{W}$ would consist of definite chain-like structures so that for any element, it would be clear which chain it belongs to. A chain such as $\cdots < y_{i-1} < y_i < y_{i+1} < \cdots$ (which could also be finite), would be identified by the fact that the distance between adjacent elements, that is the number of essential elements in the set $y_{i+1\downarrow} \setminus y_{i\downarrow}$ is small for each i.

So if we assume that a typical poset in $\mathfrak{W}$ consists of closely-packed chains then another consideration comes into effect, which again is contrary to the conclusion which was drawn in the last section, namely that it is improbable that a pair (x, y) with $x \| y$ and $x_{\downarrow} \subset y_{\downarrow}$ is close together. For let us say that we have two distinct chains, $\cdots < x_{i-1} < x_i < x_{i+1} < \cdots$ and $\cdots < y_{j-1} < y_j < y_{j+1} < \cdots$ in the poset X. If the chains are far apart, then given some element x_i of the first chain, we find that there are many elements $y_j, y_{j+1}, \ldots, y_{j+n}$ of the second chain with $x_i \| y_k$, for $k = j, \ldots, j+n$. Thus we have many pairs of unrelated elements, and each such pair could be associated with some position. On the other hand, if the two chains are close together, then there are fewer unrelated pairs between the chains, thus fewer positions which are associated with elements of both chains. Thus it will be probable that the two chains are close together, and this leads to the idea that *generalized chains* of some order n are probable. That is, chains of the form $\ldots, x_{i-1}, x_i, x_{i+1}, \ldots$, with $x_i < x_{i+n}$ for all i, yet $x_i \| x_j$ if $|i-j| < n$.

But now the question arises as to whether even these generalized chains will become packed together as closely as possible. However at this point another effect, involving the overall geometry of the poset, may become important. To illustrate this, consider three dimensional Euclidean space $\mathbb{R}^3$, with the partial ordering $(x_1, y_1, t_1) \leq (x_2, y_2, t_2)$ precisely when $t_1 \leq t_2$ and $x_1^2 + y_1^2 \leq x_2^2 + y_2^2$. If we then consider four separate "time-like" lines, say L_1, L_2, L_3 and L_4 in $\mathbb{R}^3$, and if we restrict ourselves to positions of the form $(p_\Downarrow, p_\Uparrow)$, for points $p \in \mathbb{R}^3$, then we find that if the intersections of the surface of the cone $p_\Downarrow$ with three of the lines, say L_1, L_2 and L_3 are given, then there is no further freedom in our choice of the point of intersection with L_4. Analogously, taking chains in our posets rather than time-like lines, and assuming that the posets have some definite dimension, we would expect that it would not necessarily be probable for these generalized chains to be close together rather than far apart.

All of this leads to further speculation on possible structures which would be probable in posets in $\mathfrak{W}$. In the previous paper *Discrete Partially Ordered Sets in Physics*, we have argued that structures might arise which model phenomena occurring in physics.