Poset Dimension: Various Definitions

by Geoffrey Hemion

1 Simplicial orderings

Let X be a finite partially ordered set (poset). The order '$\preceq$' can be taken to be a subset of the product $X \times X$ satisfying the usual conditions: $(a, a) \in \preceq$, for all $a \in X$; $(a, b) \in \preceq$ and $(b, a) \in \preceq$ implies $a = b$; and $(a, b) \in \preceq$ and $(b, c) \in \preceq$ implies $(a, c) \in \preceq$. A *linear order* $\preceq$ is such that for any two elements a, $b \in X$ we have either $(a, b) \in \preceq$, or $(b, a) \in \preceq$. Rather than writing '$(a, b) \in \preceq$', it is more usual to write '$a \leq b$'. If we write $a < b$, then that implies that $a \neq b$. The standard notation $a \| b$ means both $a \not\leq b$ and $b \not\leq a$. Also $a \perp b$ means that either $a \leq b$ or $b \leq a$.

Given two possible partial orders $\preceq_1$, $\preceq_2 \subset X \times X$, then if $\preceq_1 \subset \preceq_2$, we say that $\preceq_2$ is an *extension* of $\preceq_1$. It is not difficult to see that for any partial ordering of a finite set X, there exists a linear extension; that is, an extension which is a linear order. Given a finite poset $(X, \preceq)$, then there exists a finite set $\{\preceq_1, \ldots, \preceq_n\}$ of linear extensions of $\preceq$ which *realizes* $\preceq$. That means that for any pair of elements a and b with $a \| b$, there is some $\preceq_i$ with $(a, b) \in \preceq_i$, and furthermore, there is some j with $(b, a) \in \preceq_j$. Another way to describe this is to say that the original partial order $\preceq$ is the intersection $\preceq_1 \cap \cdots \cap \preceq_n$ of all the linear orders in the realizer. Finally, the *dimension* of the poset $(X, \preceq)$ is the smallest number n such that there exists a realizer with only n elements. Let us call this the *standard definition* of dimension.

But the usual idea of dimension is something which is based on geometry, not combinatorics. Let $\mathbb{R}^n$ be Euclidean n-dimensional space. A typical point in $\mathbb{R}^n$ can be represented by an n-tuple of real numbers $(x_1, \ldots, x_n)$. A method of ordering the points of $\mathbb{R}^n$ is to say that if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two points of $\mathbb{R}^n$, then $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if, and only if, $x_i \leq y_i$, for all $i = 1, \ldots, n$. Let us call this the *standard ordering* of $\mathbb{R}^n$.

A finite poset $(X, \preceq)$ has dimension n if it can be embedded in $\mathbb{R}^n$ in such a way that the ordering is preserved. And there is no such embedding in $\mathbb{R}^n$ for any $m < n$.

To see this, let us number the elements of X in some arbitrary way $\{x_1, \ldots, x_p\}$, where we assume that X contains p elements. For simplicity we could simply identify them with the first p positive integers $\{1, \ldots, p\}$. Then each linear order $\preceq_i$ in a realizer $\{\preceq_1, \ldots, \preceq_n\}$ can be represented by a permutation $(\sigma_i(1), \ldots, \sigma_i(p))$ of the set $\{1, \ldots, p\}$. Given this, then the embedding $X \to \mathbb{R}^n$ such that $x_j \to (\sigma_1(j), \ldots, \sigma_n(j))$, for each $x_j \in X$, is order-preserving. On the other hand, if there is an order preserving mapping $X \to \mathbb{R}^m$, then there must be a realizer for $\preceq$, containing m linear orders.

So given a finite poset $(X, \preceq)$ whose dimension, according to the standard definition, is n, then we see that it can be realized by an order-preserving embedding $f : X \to \mathbb{R}^n$ in $\mathbb{R}^n$, such that all elements of X are mapped to points of $\mathbb{R}^n$, all of whose coordinates are positive. By perhaps adjusting the coordinates slightly, we can assume that the embedding is such that no two elements of X have any coordinates which are equal.

Concentrating on the Euclidean space $\mathbb{R}^n$, let $(q_1, \ldots, q_n) \in \mathbb{R}^n$ be a typical point. Let $H \subset \mathbb{R}^n$ be the $n - 1$ dimensional hyperplane given by the equation $q_1 + \cdots + q_n = 0$. For each element $x_j \in X$ in its embedding in $\mathbb{R}^n$, let $\Delta_j \subset H$ be the set of points of H which are less than $f(x_j)$, in the standard ordering of $\mathbb{R}^n$. Thus Δ_j has the natural structure of an $(n-1)$-simplex, and furthermore, given any two such $(n-1)$-simplexes Δ_j and Δ_k, we see that their corresponding faces must be parallel. Also, by perhaps adjusting slightly the coordinates assigned to the elements of X, we may assume that all of these simplexes are in general position with respect to one another. Identifying H with $\mathbb{R}^{n-1}$, we have a representation of the poset $(X, \preceq)$ in terms of a set of $(n-1)$-simplexes in $\mathbb{R}^{n-1}$, with corresponding faces parallel, such that $x_j \leq x_k$ if, and only if, $\Delta_j \subseteq \Delta_k$.

Definition 1. *A representation of a finite poset X in terms of a set of n-simplexes in general position in some Euclidean space $\mathbb{R}^n$, such that corresponding faces are always parallel, and such that the ordering of X is given by the inclusion ordering of the simplexes, will be called a* simplicial ordering *on X.*

Therefore we see that a finite poset $(X, \preceq)$ has dimension n, according to the standard definition, if and only if it has an $(n-1)$-dimensional simplicial ordering, and no simplicial ordering of smaller dimension.

Definition 2. *A poset $(X, \preceq)$ which has dimension n according to the standard definition of dimension, will also be said to have* simplicial dimension *n. That is, there exists a representation of X in terms of parallel simplexes in $\mathbb{R}^{n-1}$, but there is no such representation in $\mathbb{R}^{n-2}$.*

2 Sphere orderings

A sphere in $\mathbb{R}^n$ with center $\mathbf{x} \in \mathbb{R}^n$ and radius $r \geq 0$ is the boundary of the spherical ball $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \leq r\}$. Given a set of such spherical balls $B_1, \ldots, B_p$ in $\mathbb{R}^n$, then we can assign a partial ordering to the set by the rule $B_i \leq B_j$ if and only if $B_i \subseteq B_j$. That is, the inclusion ordering. Given a partially ordered set $(X, \preceq)$, then we will say that it has a *sphere order* of dimension n if there is an isomorphism (that is, an order-preserving bijection) onto a set of spherical balls $B_1, \ldots, B_p$ in $\mathbb{R}^{n-1}$.

Rather than concentrating on the interiors of the B_i, it is more convenient to concentrate on the boundaries ∂B_i. Such a boundary is an $(n-1)$-sphere, which we denote by S_i. There is no loss of generality if we assume that if $S_i \cap S_j \neq \emptyset$, then $S_i \cap S_j$ is an $(n-2)$-sphere in $\mathbb{R}^n$. And more generally, we may assume that the intersection of any m of the $(n-1)$-spheres, $S_{i(1)} \cap \cdots \cap S_{i(m)}$, for $m < n-1$, is either empty, or else it is an $(n-m-1)$-sphere. That is, the set of spheres $S_1, \ldots, S_p$ is in *general position* in $\mathbb{R}^n$.

Definition 3. *The finite poset $(X, \preceq)$ will be said to have the* sphere dimension n *if it has a sphere order in terms of spherical balls in $\mathbb{R}^{n-1}$, but there is no such representation in $\mathbb{R}^{n-2}$.*

3 Relationships between simplicial dimension and sphere dimension

What is the relationship between the simplicial dimension and the sphere dimension? Surprisingly little!

Let Q_n be the "standard n-dimensional poset". That is, Q_n consists of $2n$ elements, $Q_n = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ such that for each i, we have $a_i \| b_i$, and $a_i < b_j$ for $i \neq j$. Apart from that we have $a_i \| a_j$ and $b_i \| b_j$, for all $i \neq j$.

Then, as the name implies, Q_n has dimension n, according to the standard definition of dimension.[1] Hence the simplicial dimension of Q_n is also n. However, as is well known, the sphere dimension of Q_n is 3, for all $n \geq 3$. Thus there are posets with large simplicial dimension, yet small sphere dimension.

On the other hand, Felsner, Fishburn, and Trotter[2] have proven the remarkable result that there exists a finite poset with simplicial dimension 3,

[1] For example, see the book "*Combinatorics and Partially ordered Sets*", by William T. Trotter.

[2] *Finite Three Dimensional Partial Orders which are not Sphere Orders*, 1998.

yet no sphere order of any dimension. So one could say that in some sense, this finite poset has infinite sphere dimension!

Somehow though, one feels that these results simply reflect the limitations which arise when we insist on retaining the rigidity of spheres at all costs. After all, if we are allowed to "bend" the spheres, then one kind of bending allows us to transform spheres into simplexes, then back again. Wouldn't a more sensible "topological" definition be possible, encompassing both possibilities?

4 How spheres intersect

Theorem 1. *Let S be an $(n+1)$-dimensional sphere (or simply $\mathbb{R}^{n+1}$), $n > 1$, and let S_1, S_2, and S_3 be three n-dimensional spheres embedded in S in general position. Assume that both $S_1 \cap S_3$ and $S_2 \cap S_3$ are $(n-1)$-dimensional spheres on S_3, and $(S_1 \cap S_3) \cap (S_2 \cap S_3) = \emptyset$. For each i, let B_i be the closure of one of the components of $S \setminus S_i$ $(i = 1, 2, 3)$. If we take $\mathbb{R}^{n+1}$ rather then the $(n+1)$-sphere S, then choose B_i to be the component which does not include the point at infinity. There are then $2^3 = 8$ possibilities for whether points of S are, or are not, in the various sets B_1, B_2, and B_3. However, at least one of these possibilities is empty.*

Proof. The two $(n-1)$-spheres $S_1 \cap S_3$ and $S_2 \cap S_3$ are disjoint on S_3. Therefore they separate S_3 into three components. If S_1 and S_2 intersect within B_3 then they separate B_3 into four components. However in this case, S_1 and S_2 cannot intersect within $S \setminus B_3$. Therefore S_1 and S_2 separate $S \setminus B_3$ into only three components. □

Therefore we conclude that if we have such a situation as described in theorem 1: an $(n+1)$-dimensional sphere S, containing three n-dimensional spheres in general position, then if all 8 possible combinations of being on one side or the other of the various n-spheres are realized by non-empty subsets of S, it must follow that the intersection of the three n-dimensional spheres is, in fact, an $(n-2)$-dimensional sphere.

5 Spheres on the surface of a simplex

Theorem 2. *Let Δ be an n-simplex, and let $C_1, \ldots, C_m$ (with $m \leq n$) be a collection of $(n-1)$-dimensional cells in Δ. That is to say, each C_j is a union of some subset of the $(n-1)$-dimensional sub-simplexes (or faces) of Δ. Furthermore, we assume that $C_i \cap C_j$ is at most an $(n-2)$-dimensional* *complex for each pair $i \neq j$. i.e. the C_j have no "overlap". Finally, we assume that the $(n-1)$-cells cover $\partial\Delta$, the boundary of Δ. That is, every $(n-1)$-sub-simplex of Δ is contained in one, and only one, of the C_j. Then $C_1 \cap \cdots \cap C_m$ is a combinatorial $(n-m-1)$-sphere.*

Proof. If $m = n$ then one single $(n-1)$-cell, say C_1, consists of two of the $(n-1)$-simplexes (faces) of Δ. All the rest of the C_j only consist of a single face. Therefore in this case $C_1 \cap \cdots \cap C_m$ consists of just two of the vertexes of Δ. Thus it is a combinatorial 0-simplex, as required.

If $m < n$ we use induction on the number $n - m$. But to begin, note that since each face of Δ contains n vertexes, while Δ itself contains $n+1$ vertexes, it follows that the intersection of any two different faces is an $(n-2)$-simplex containing $n-1$ vertexes. Similarly, if α is an $(n-2)$-simplex in Δ, and δ is an $(n-1)$-simplex (that is to say, a face of Δ), such that α is not contained in δ, then $\alpha \cap \delta$ must be an $(n-3)$-simplex on the boundary of α.

The next thing to do is to prove that $C_1 \cap \cdots \cap C_m \neq \emptyset$. In particular, we prove that there is some vertex v of Δ in $C_1 \cap \cdots \cap C_m$. Take some vertex w which is not in all the C_j. (Obviously, if no such w exists then we are trivially finished.) Say, $w \notin C_1$. Then C_1 must be the single face of Δ which doesn't contain w. Therefore, all other faces of Δ must meet C_1 in $(n-2)$-simplexes on the boundary of C_1. These faces belong to the C_j, for $j > 1$. They give us $m-1$ collections of $(n-2)$-simplexes on the single $(n-1)$-simplex C_1. An induction proves that $C_1 \cap \cdots \cap C_m \neq \emptyset$.

So let v be a vertex in $C_1 \cap \cdots \cap C_m$. Let δ be the face of Δ which doesn't contain v. We must have δ being contained in one of the C_j; say δ is in C_1. Let us say that C_1^* is the set of faces of Δ in C_1, but not including δ.

The boundary, $\partial\delta$ consists of n simplexes, each of dimension $(n-2)$. The intersections $C_j \cap \partial\delta$, for $j > 1$, together with $C_1^* \cap \partial\delta$, give us m collections of faces, call them $c_1, \ldots, c_n$, of the $(n-1)$-simplex δ. Therefore, an induction proves that $c_1 \cap \cdots \cap c_m$ is an $((n-1)-m-1)$-dimensional combinatorial sphere S_v on $\partial\delta$. Connecting each of the $((n-1)-m-1)$-simplexes of S_v to the vertex v gives us a combinatorial $(n-m-1)$-dimensional disc D which is contained in $C_1 \cap \cdots \cap C_m$.

Therefore we know that $C_1 \cap \cdots \cap C_m$ contains more than one vertex. Let $v_1 \neq v$ be another vertex in $C_1 \cap \cdots \cap C_m$. Let δ_1 be the face of Δ which does not contain v_1. Since $v_1 \in C_1 \cap \cdots \cap C_m$, it cannot be the case that δ_1 is by itself one of the C_j. Therefore, arguing as before, we obtain another combinatorial sphere S_{v_1} on the boundary of δ_1, such that S_{v_1} is contained in $C_1 \cap \cdots \cap C_m$. However, the sphere S_{v_1} must be contained within D. Thus it bounds a combinatorial disc D_1 in D. As before, we connect the simplexes of S_{v_1} to v_1, giving us a further combinatorial disc D_2. The union of D_1 and

D_2 is then the sought-after $(n-m-1)$-sphere which is $C_1 \cap \cdots \cap C_m$. $\square$

6 Complete posets and the set $P_*(Q_n)$

Given a finite poset $(X, \preceq)$, we can consider the set of *positions* which are contained within the poset. A position is a pair of non-empty subsets of X, namely (U, V), such that $u \leq v$ for all $u \in U$ and $v \in V$. The pair is assumed to be *maximal*, in the sense that if $a \in X$ with $a \leq v$ for all $v \in V$, then $a \in U$; also if $b \in X$ with $b \geq u$ for all $u \in U$, then $b \in V$. We can say that U is the *lower set* of the position, and V is the *upper set* of the position.

Let $P(X)$ be the set of positions of $(X, \preceq)$. Then $P(X)$ is itself a poset in a natural way, and it contains X as a sub-poset. It is complete in the sense that $P(P(X))$ is simply isomorphic to $P(X)$. All of this is related to the well-known *MacNeille completion* of posets.

It is interesting to consider $P(Q_n)$. This has (nearly) the combinatorial structure of an $(n-1)$-simplex. The vertexes of this simplex are the positions whose lower sets contain just one of the elements $\{a_i\}$. Thus there are n different vertexes. The edges are the positions whose lower sets consist of precisely two of the elements $\{a_i, a_j\}$, where $i \neq j$. The 2-dimensional faces are the positions whose lower sets consist of three distinct elements from the set $\{a_1, \dots, a_n\}$, and so forth. But note that there is no position corresponding with the whole $(n-1)$-simplex itself, since the whole set $\{a_1, \dots, a_n\}$ cannot be the lower set of a position in $P(Q_n)$. (For otherwise, the corresponding upper set would be empty.) Therefore, $P(Q_n)$ has the structure of the *boundary* of a combinatorial $(n-1)$-simplex. Put another way, $P(Q_n)$ is a combinatorial $(n-2)$-sphere. The MacNeille completion involves adding in a single element below all other elements in $P(Q_n)$, and also an element above all other elements in $P(Q_n)$. Let us call the resulting poset $P_*(Q_n)$. It is a lattice; the smallest lattice containing Q_n.

Given any finite poset $(X, \preceq)$ with simplicial dimension n, then the simplicial dimension of $P(X)$ is also n. Thus the simplicial dimension of $P(Q_n)$ is n. On the other hand we have seen above that the sphere dimension of Q_n is only 3, for $n \geq 3$. But then we have the following theorem[3]:

Theorem 3. *The sphere dimension of $P_*(Q_n)$ is at least n.*

Proof. To see this, let us assume that $P_*(Q_n)$ has a sphere order which is less than n, and we then look for a contradiction. Therefore we begin by assuming that there is a collection of spherical balls in the Euclidean space $\mathbb{R}^{n-2}$ which,

[3]These results were proven in: *Posets und Positionen*, by Joerg Zender (Diplomarbeit, Universität Bielefeld, 2008

when considered to be partially ordered by set inclusion, is isomorphic with $P_*(Q_n)$. The boundaries of the balls are $(n-3)$-spheres, and we assume that they are in general position.

In particular, if we disregard the greatest element of $P_*(Q_n)$, we see that there are n maximal elements of Q_n, call them $x_1, \ldots, x_n$, which are also maximal elements of $P(Q_n)$. These n maximal elements correspond with n spherical balls $B_1, \ldots, B_n$ in $\mathbb{R}^{n-2}$, representing the $(n-1)$-dimensional sphere order for $P_*(Q_n)$. The spheres which are the boundaries of the spherical balls will be called $S_1, \ldots, S_n$. They are all spheres of dimension $n-3$, embedded in general position in $\mathbb{R}^{n-2}$.

Since the $x_1, \ldots, x_n$ are all unrelated to one another, we have $B_i \not\subset B_j$, for all $i \neq j$. But also, given any $m \leq n$ of the x_i, say $x_{i(1)}, \ldots, x_{i(m)}$, then there is an element of $P_*(Q_n)$ which is beneath just these elements, but not beneath any of the other x_j. Thus in $\mathbb{R}^{n-2}$, there is a spherical ball in the sphere order for $P_*(Q_n)$ which is contained in $B_{i(1)} \cap \cdots \cap B_{i(m)}$, yet not contained in any of the other B_j.

Let us now consider the sphere S_n, and we examine the intersections $B_i \cap S_n$, for $i = 1, \ldots, n-1$. These are $(n-3)$-dimensional cells on S_n, each of whose boundaries is an $(n-4)$-sphere. (Since we must have $B_1 \cap \cdots \cap B_n \neq \emptyset$, it follows that each of the sets $B_i \cap S_n$ is non-empty.) Let us write $C_i = B_i \cap S_n$, for each i.

Can it be that $C_i \subset C_j$, for some $i \neq j$? That would mean that the boundary of C_i would have no intersection with the boundary of C_j. But that would contradict theorem 1, since in $\mathbb{R}^{n-2}$, all combinations of possibilities for points being either within, or not within B_i, B_j or B_n, must represent non-empty subsets of $\mathbb{R}^{n-2}$.

In addition to the C_i, we can also consider the intersections of the other spherical balls with S_n in the sphere order for $P_*(Q_n)$. Obviously, for the balls contained within B_n, representing the elements of $P_*(Q_n)$ beneath x_n, the intersection with S_n is empty. The balls representing the elements of $P_*(Q_n)$ not beneath x_n must all intersect S_n (since they all contain the lowest element of $P_*(Q_n)$). So we have a system of $(n-3)$-cells on S_n in general position. For these cells, theorem 1 shows that given two cells on S_n, one is contained within the other if and only if the corresponding spherical ball in $\mathbb{R}^{n-2}$ is contained within the other.

As before, we can take any $m \leq n-1$, and choose any combination $C_{i(1)}, \ldots, C_{i(m)}$ of the $n-1$ cells $C_1, \ldots, C_{n-1}$ on S_n. Then there is a cell on S_n which is contained in $C_{i(1)} \cap \cdots \cap C_{i(m)}$, yet not contained within any of the other C_j.

Repeating this procedure, reducing the number of cells and the dimension at each step, we finally arrive at the situation that we have four 2-cells in general position in the two-sphere. In addition to these, we must have further 2-cells, representing elements of $P_*(Q_n)$ such that for any combination of the four 2-cells, a cell is contained within their intersection, but not contained in any of the other of the four 2-cells. This is impossible. $\square$

7 The same proof with simplexes

Of course we already know that the simplicial dimension of $P_*(Q_n)$ is n. However we would like to see if the proof in the previous section is also valid when sets of $(n-1)$-simplexes with parallel faces in $\mathbb{R}^{n-1}$ are used instead of the spherical balls in that proof. So what were the properties of those spherical balls which were needed?

- To begin with, it was assumed that the spheres, $S_1, \ldots, S_p$, which were the boundaries of the balls, $B_1, \ldots, B_p$, representing all of the elements in $P_*(Q_n)$, were in general position.

  Similarly, if $\Delta_1, \ldots, \Delta_p$ are the simplexes in $\mathbb{R}^{n-1}$ in a given simplicial ordering of $P_*(Q_n)$, then we may assume that the boundaries $\partial\Delta_1, \ldots, \partial\Delta_p$ are in general position.

- Any non-empty intersection of m of the spheres $S_{i(1)} \cap \cdots \cap S_{i(m)}$ in $\mathbb{R}^{n-1}$ is a single $(n-m-1)$-sphere.

  Similarly, if $\Delta_{i(1)} \cap \cdots \cap \Delta_{i(m)}$ is not empty, then it is itself a single $(n-1)$-simplex, Δ, in $\mathbb{R}^{n-1}$. The set $\partial\Delta_{i(1)} \cap \cdots \cap \partial\Delta_{i(m)}$ is contained in $\partial\Delta$. More specifically, each of the sets $\partial\Delta_{i(j)}$ is a combinatorial $(n-2)$-cell on $\partial\Delta$. Therefore, according to theorem 2, $\partial\Delta_{i(1)} \cap \cdots \cap \partial\Delta_{i(m)}$ must be a combinatorial $(n-m-1)$-sphere on $\partial\Delta$.

- The proof of theorem 3 only uses the topological relationships involved in the embeddings of the various spheres. Therefore it applies equally well to the boundaries of the simplexes in a simplicial ordering.

8 Cell orderings

Given a simplicial ordering of a finite poset $(X, \preceq)$, then each of the elements of X is represented by a simplex in $\mathbb{R}^{n-1}$. But of course each such simplex is just the simplest kind of combinatorial $(n-1)$-cell. On the other hand, if we have a sphere ordering of the poset, then the elements are represented by spherical balls in $\mathbb{R}^{n-1}$ whose boundaries are $(n-2)$-spheres. Yet it is a simple matter to replace these spherical balls with simplicial $(n-1)$-cells, say $G_1, \ldots, G_p$, each of which is a simplicial complex, whose boundaries $\partial G_1, \ldots, \partial G_p$ are simplicial $(n-2)$-spheres. This can be done in such a way that any non-empty intersection of m of the boundaries $\partial G_{i(1)} \cap \cdots \cap \partial G_{i(m)}$ is a single combinatorial $(n-m-1)$-sphere.

Let us call any such representation of a finite poset in terms of $(n-1)$-cells in $\mathbb{R}^{n-1}$ a *cell ordering* of dimension n.

Definition 4. *The finite poset $(X, \preceq)$ will be said to have* cell dimension *n if there exists a cell ordering of dimension n, and n is the smallest such number.*

Therefore we see that the cell dimension is always no greater than either the simplicial (or standard) dimension, or the sphere dimension. Furthermore, the proof of theorem 3 can be equally applied to cell orders to show that the cell dimension of $P_*(Q_n)$ is at least n. However, it cannot be more than n, since the simplicial dimension is n. This leads to the theorem:

Theorem 4. *The cell dimension of $P_*(Q_n)$ is n.*

And more generally, we can say that if a poset $(X, \preceq)$ contains $P_*(Q_n)$ as a subset, then its cell dimension is at least n. On the other hand, Q_n alone only has cell dimension 3, so it may be that a poset of high dimension, according to the standard definition, only has a low cell dimension. Yet we can certainly say that any finite poset does have a finite cell dimension.

In a given poset $(X, \preceq)$, a totally ordered subset $C \subset X$ is called a *chain*. A subset $A \subset X$ which is such that for all x, $y \in A$ we have $x \| y$, is called an *antichain*. A chain, or an antichain is *maximal* if it is not properly contained within a further chain, or antichain, respectively. If we are interested in describing the phenomena of the physical world in terms of posets, then it is natural to associate maximal chains with "time lines", and maximal antichains would be space-like hyperplanes. That is to say, we have the association: chains $\longleftrightarrow$ time; antichains $\longleftrightarrow$ space.

Time is only one dimensional, and thus it is presumably devoid of geometry. It is in space that we expect to experience geometric phenomena, and therefore it is natural to concentrate on maximal antichains. Given such an antichain A, then the elements x of X above A can be represented (not necessarily uniquely) by the set $A(x) = \{a \in A : a < x\}$. In Minkowski space, if A is a hyperplane then the sets of the form $A(x)$ are spherical balls, and this was the motivation for defining the sphere dimension of an arbitrary finite poset. But according to the theory of relativity, physical space is not flat, and thus the sets $A(x)$ are no longer spherical balls. Hence the idea of the cell dimension would seem to be the more appropriate definition for use in physics.[4]

[4] Note in this connection that we have shown in
" www.math.uni-bielefeld.de/~hemion/local_probabilities.pdf "
that local 4-dimensionality is a natural characteristic of finite partially ordered sets.