Local Probabilities in Finite Partially Ordered Sets

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1 Definitions

Let X be a partially ordered set, or poset. For elements $x, y \in X$ we distinguish the two cases that x and y are either related to one another or they are unrelated. We use the notation $x \perp y$ to mean that either $x \leq y$ or $y \leq x$, and $x \| y$ means that both $x \not\leq y$ and $y \not\leq x$. Given $x < y$ then the set of elements between x and y is

$$x \between y = \{u \in X : x < u < y\}.$$

The set of elements beneath x is

$$x_{\downarrow} = \{u \in X : u < x\}$$

and if the element x is to be included in this set then we have

$$x_{\Downarrow} = \{u \in X : u \leq x\}.$$

The sets $x_{\uparrow}$ and $x_{\Uparrow}$ are defined analogously for elements above x.

As a first idea of what we mean by local probabilities, consider the following. Let n and N be two numbers with $n < N$. We then take all possible partially ordered sets X consisting of N elements, and for each such X we consider all possible pairs $a < b$ of elements in X such that $a \between b$ consists of precisely n elements. These different sets of the form $a \between b$ fall into equivalence classes, where two such sets are equivalent if they are isomorphic as partially ordered sets. Then we can say that the relative probabilities of the different equivalence classes of these "between sets" is proportional to the number of sets in each class. Given n, the relative probabilities will depend upon the choice of N. Perhaps in the limit as N becomes larger, these relative probabilities might converge to some limiting values.

But there is a problem with this method. To illustrate the problem, consider the situation with $n = 2$. That is, we have two elements $a < b$ and two further elements u and v with $a < u < b$ and $a < v < b$. There are two cases: either $u \| v$ or $u \perp v$. In the case $u \perp v$, we either have $u < v$ or else $v < u$. But both are isomorphic as partially ordered sets, so to be definite we choose $u < v$. Let U and U' each be posets consisting of the elements $\{a, b, u, v\}$, with U having $u \| v$ and U' having $u < v$.

For some larger N, let X be a poset containing as a subposet U, and we assume that also in X we have $a \between b = \{u, v\}$. Now take some arbitrary $x \in X$, not in U. For each of the elements u and v we might have x either being related to them, or not. There are seven possibilities: either x is unrelated to both u and v, or x is either above or below one of them, or x is either above or below both.

On the other hand, analogously, if X' contains U' then there are only five possibilities for the relationships an element $x' \in X'$ which is not in U' can have with $\{u, v\}$. Either x' is unrelated to both u and v, or $x' > v$ (and thus automatically $x' > u$), or $x' > u$ while $x' \not> v$, or $x' < u$ (thus $x' < v$), or $x' < v$ while $x' \not< u$.

This shows that we will generally expect to have more copies of U in a typical X than there are copies of U' in a typical X'. That is, the idea of simply counting the number of elements in U and comparing it to the number of elements in U' does not produce a fair comparison.[1] Instead we should take the sets $u_{\Downarrow}$, $v_{\Downarrow}$, and $u_{\Uparrow}$, $v_{\Uparrow}$, since they determine possible ordering relations with other elements of larger posets. For U we have the possibilities $\emptyset$, $u_{\Downarrow}$, $v_{\Downarrow}$ and $u_{\Downarrow} \cup v_{\Downarrow}$ for an element x to be either unrelated, or above either one or both of u or v. That is, we are interested in the power set of $\{u_{\Downarrow}, v_{\Downarrow}\}$, call it $\mathcal{P}(u_{\Downarrow}, v_{\Downarrow})$. It has four elements. As for U', the power set $\mathcal{P}(u_{\Downarrow}, v_{\Downarrow})$ will only contain three elements, namely $\emptyset$, $u_{\Downarrow}$ and $v_{\Downarrow}$. We must similarly examine the power sets $\mathcal{P}(u_{\Uparrow}, v_{\Uparrow})$. It is these power sets which are relevant when comparing local variations within a given, large poset to one another. And thus we see that on this basis, the sets U and U' are incomparable.

Therefore, in our example, U' can only be compared with itself. On the other hand, U can be compared with the poset U'' which contains the three elements u, v and w with $u < v < w$, since in this case the power sets $\mathcal{P}(u_{\Downarrow}, v_{\Downarrow}, w_{\Downarrow})$ and $\mathcal{P}(u_{\Uparrow}, v_{\Uparrow}, w_{\Uparrow})$ each contain four sets. Let us call these the lower and upper power sets of U.

More generally, given two numbers n_1 and n_2, we consider all possible posets U having some number m of elements $\{u_1, \dots, u_m\}$ such that the lower power set $\mathcal{P}(u_{1\Downarrow}, \dots, u_{m\Downarrow})$ consists of n_1 sets and the upper power set

[1] An example of an unfair comparison would be to take the set of all posets with either 2 or 3 elements. The majority contain 3 elements. Is it then reasonable to conclude that 3 is more probable than 2? Surely not, since each of the posets with 3 elements contains various instances of subposets with 2 elements.

$\mathcal{P}(u_{1\Uparrow}, \ldots, u_{m\Uparrow})$ consists of n_2 sets. Then all such U will be taken to be equally probable. The set of all possible local variations in a given region will then fall into different equivalence classes characterized by the different numbers n_1 and n_2. Relative probabilities only make sense *within* the different equivalence classes.

This definition involves a number of assumptions. It is assumed that any larger poset X containing such subposets as U contains no elements "within" U. That is, there are no $x \in X$, not in U, such that both $x_{\Downarrow} \cap U \neq \emptyset$ and $x_{\Uparrow} \cap U \neq \emptyset$. Or put another way, U is a *local region of variation* within X. And then, of course, we assume that the details of any containing poset X play no further role in the calculation of these probabilities.

The examples we will consider will be symmetrical with respect to changing "less than" to "greater than". And so for simplicity we only consider the lower power sets. This leads to the definition:

Definition. *Let n be a given number. Considered as variations of a region of a poset X, all subposets U whose lower power sets consist of n sets will be taken to be equally probable.*

In what follows we will generally consider a region $a \between b$ of some generic X and we will call the lower power set of the set of elements of a configuration between a and b simply *the power set* of that configuration.

2 Generalized chains

Let some region $a \between b$ be given and let us consider the case that $a \between b$ consists of n unrelated elements. That is

$$a \between b = U = \{u_1, \ldots, u_n\}$$

such that for any $i \neq j$ we have $u_i \| u_j$. Then the power set consists of 2^n subsets.

Another possible configuration for $a \between b$ might be a chain of length $2^n - 1$. That is

$$a \between b = V = \{v_1, \ldots, v_{2^n-1}\}$$

with $v_i < v_{i+1}$ for all $1 \leq i \leq 2^n - 2$. Again, the power set consists of 2^n subsets in this configuration, so that it has the same probability as does U.

A third possibility for $a \between b$ consists of $2^n - 2$ elements which form a simple *generalized* chain, namely

$$a \between b = W_1 = \{w_1, \ldots, w_{2^n-2}\}$$

with $w_1 \| w_2$, $w_1 < w_3$, $w_2 < w_3$, and then $w_i < w_{i+1}$ for all $3 \leq i \leq 2^n - 3$. Once again the size of the power set is 2^n so that W_1 has the same probability as the other two configurations which we have considered so far.

But then we also have

$$a \between b = W_2 = \{w_1, \ldots, w_{2^n-2}\}$$

with $w_2 \| w_3$, $w_1 < w_2$, $w_1 < w_3$, $w_2 < w_4$, $w_3 < w_4$, and then $w_i < w_{i+1}$ for all $4 \leq i \leq 2^n - 3$. Again, the size of the power set is 2^n.

In fact, following this pattern one sees that we have $2^n - 3$ such generalized chains $\mathcal{W}_1, \ldots, \mathcal{W}_{2^n-3}$, all of whose power sets have 2^n subsets, and thus they are all equally probable.

Many further configurations fit into this pattern. For example we have generalized chains with two pairs of unrelated elements: $w_i \| w_{i+1}$ and $w_j \| w_{j+1}$ for $|i - j| \geq 2$ and the rest of the elements forming a simple chain. If $i = j - 2$ we might also consider $w_i < w_j$, while $w_{i+1} \not< w_j$, and so on. There are nearly n^2 such configurations. And then we can have generalized chains with three pairs of unrelated elements. There are nearly n^3 such configurations, although we do notice that the number of elements in the configuration is reduced by one for each such addition.

Once we have exhausted all these possibilities we can then consider generalized chains having various numbers of unrelated triples, quadruples, and so forth. Eventually we come back to our single original set $\mathcal{V}$ consisting of n unrelated elements. All of these configurations are equally probable.

If we just restrict ourselves to the types of configurations which have been described here, it is obvious that if the number n is reasonably large and a configuration U for the set $a \between b$ is chosen at random, then we expect the height of U to be much greater than its width. Here the height is taken to be the length of the largest possible simple chain, and the width is the number of elements in the largest possible subset consisting of mutually unrelated elements. Therefore U probably resembles a chain; a fuzzy, or generalized chain.

3 Other structures

The generalized chains considered in the last section are characterized by having a limited width such that for each element u in the chain, the number of elements unrelated to u is limited. If the number of elements we are considering is much greater than these limits then other structures than a single generalized chain might be probable.

As an example let us consider two simple chains, of height k and l:

$$U = \{u_1, \ldots, u_k : u_i < u_{i+1}, \forall 1 \le i < k\},$$

$$V = \{v_1, \ldots, v_l : v_i < v_{i+1}, \forall 1 \le i < l\}.$$

If $u_i \| v_j$ for all i and j then the number of subsets in the power set is $(k+1) \times (l+1)$. Thus if we are to compare this with a single chain, that chain would have to have height $(k+1) \times (l+1) - 1$. But in general there may be some relations of the form $u_i < v_j$ or $v_i < u_j$. How many possible different relations and how many subsets can there be?

To begin to estimate how many different possible relations there are, let us consider the possibilities for relations of the form $u_i < v_j$. The thing to note is that if $u_i < v_j$ then we must also have $u_{i-1} < v_j$ and also $u_i < v_{j+1}$. Let is write $F(k, l)$ to represent the number of possible relations here. Then we see that if $u_k < v_l$, it is clear that the number of possible relations for the further $l-1$ elements of V is given by $F(k, l-1)$. If $u_k \not< v_l$, but $u_{k-1} < v_l$ then the number of possible relations for the further $l-1$ elements of V is $F(k-1, l-1)$. Proceeding down through U, we obtain the recursive formula:

$$\begin{aligned}
F(k, l) &= \sum_{t_1=1}^{k} F(t_1, l-1) \\
&= \sum_{t_1=1}^{k} \sum_{t_2=1}^{t_1} F(t_2, l-2) \\
&\vdots \\
&= \sum_{t_1=1}^{k} \sum_{t_2=1}^{t_1} \cdots \sum_{t_{l-1}=1}^{t_{l-2}} F(t_{l-1}, 1) \\
&= \sum_{t_1=1}^{k} \sum_{t_2=1}^{t_1} \cdots \sum_{t_{l-1}=1}^{t_{l-2}} t_{l-1}
\end{aligned}$$

since $F(t_{l-1}, 1) = t_{l-1}$.

We can bound this sum from below by taking integrals.

$$\begin{aligned}
\sum_{t_1=1}^{k}\sum_{t_2=1}^{t_1}\cdots\sum_{t_{l-1}=1}^{t_{l-2}} t_{l-1} &> \int_0^k\int_0^{t_1}\cdots\int_0^{t_{l-2}} s_{t_{l-1}}\, ds_{t_{l-1}}\dots ds_2 ds_1 \\
&= \int_0^k\int_0^{t_1}\cdots\int_0^{t_{l-3}} \frac{{s_{t_{l-2}}}^2}{2}\, ds_{t_{l-2}}\dots ds_2 ds_1 \\
&= \int_0^k\int_0^{t_1}\cdots\int_0^{t_{l-4}} \frac{{s_{t_{l-3}}}^3}{3*2}\, ds_{t_{l-3}}\dots ds_2 ds_1 \\
&\vdots \\
&= \frac{k^{l-1}}{(l-1)!}
\end{aligned}$$

Stirling's formula is

$$(l-1)! \approx \sqrt{2\pi(l-1)}\left(\frac{l-1}{e}\right)^{l-1},$$

giving

$$F(k,l) \gtrapprox \frac{1}{\sqrt{2\pi(l-1)}}\left(\frac{ke}{l-1}\right)^{l-1}.$$

Assuming that we have chosen the chain U to be not shorter than V, then we have $k \geq l$ and so $F(k,l)$ grows very rapidly as the height of the chains grows. (Note that even if $l > k$, the number $F(k,l)$ will still be large despite the fact that our integral approximation will now underestimate things drastically.)

In addition to the possible relations of the form $u_i < v_j$, we also have the relations of the form $v_i < u_j$ to take into account. For each possible configuration considered in the calculation of $F(k,l)$ we have many such further relations, the number of which should be multiplied by $F(k,l)$ to obtain the total number of different configurations. But note however that we can never have a pair with both $u_i < v_j$ and at the same time $v_j < u_i$.

These different configurations generally have different numbers of subsets in their power sets so that they cannot be compared with one another. However if we consider configurations with relatively large numbers n of elements then the equivalence classes of the configurations which are comparable will generally be large. This follows since, as we have seen, there are less than n^2 possible such equivalence classes, yet many more possible configurations.

The conclusion is that if n is reasonably large, and if we restrict ourselves to configurations having either one or two simple chains, then it is overwhelmingly probable that we will have a configuration with two simple chains.

But it is also obvious that there are many more possibilities besides these. For example we might consider generalized chains whose width is limited by some fixed value w. For n large in comparison to w, any element of the generalized chain is not related to only a small number of further elements in the chain. Thus we can think of the generalized chain as consisting of a large number of segments which act somewhat as if they were elements of a simple chain. We can then compare the probability of having a single generalized chain with two interacting, but separate shorter generalized chains. We can apply our argument in this case as well to show that for large n, it is probable that we will have two separate generalized chains. Extending the argument to larger numbers of generalized chains, we see that as n increases, it is probable that we will have proportionally more of these interacting generalized chains.

Are there other structures besides generalized chains, or at least configurations constructed from pieces of generalized chains, which are probable as local structures in finite partially ordered sets? Perhaps not.

4 Dimension

As in the last section, let us assume that we have two distinct, simple chains of height p and q:

$$U = \{u_1, \ldots, u_p : u_i < u_{i+1}, \forall 1 \leq i < p\},$$

$$V = \{v_1, \ldots, v_q : v_i < v_{i+1}, \forall 1 \leq i < q\}.$$

We would like to examine a region which can be "indexed" by the elements along U and V in the following manner.

Let x be some element in this region. Then there are four unique elements, $\{u_i, u_j, v_k, v_l\}$, two along U and two along V, which index the element x. The two elements u_i and u_j on U indexing x are such that $x < u_i$ but $x \not< u_{i-1}$ and $u_j < x$ but $u_{j+1} \not< x$. The elements along V are similarly such that $x < v_k$ but $x \not< v_{k-1}$ and $v_l < x$ but $v_{l+1} \not< x$. We assume that the elements in the region are uniquely represented by this indexing. That is, if x' is some other element in the region indexed by the four elements $\{u_{i'}, u_{j'}, v_{k'}, v_{l'}\}$, then at least one of those elements is different from the indexing of x.

We are interested in the question of whether or not x and x' are related to one another. Given the way we have chosen the index elements, we see that if $x < x'$ then we must have $u_i \leq u_{i'}$, $u_j \leq u_{j'}$, $v_k \leq v_{k'}$, and $v_l \leq v_{l'}$. For all other relationships of the indexing elements, we must have $x \not< x'$. A similar condition holds for $x' \not< x$. If both conditions hold, then we must have $x \| x'$.

Therefore we must have $x \| x'$ in all cases *except* when the indexing elements of x are all less than (or they are all greater than) or equal to the respective indexing elements of x'. But if, say, the indexing elements of x are all less than or equal to the respective indexing elements of x' then the situation is unclear. We might have either $x < x'$ or $x \| x'$.

So let us consider the case that all the respective indexing elements of x are less then or equal to those of x'. We then have two different possible configurations: $W_<$ where $x < x'$, and $W_\|$ where $x \| x'$. Then the power set of $W_<$ contains fewer subsets in comparison with $W_\|$.

To see this, begin by observing that most of these subsets are the same, both for $W_<$ and for $W_\|$. Only those which contain x and/or x' might be different. So let us assume that there are m elements along U between $u_{j'}$ and u_i, and furthermore we assume that there are n elements along V between $v_{l'}$ and v_k. (If either m or n were to be zero then we must have $x < x'$, and so the case $x \| x'$ would not occur.) Each of the subsets of $W_<$ corresponds with a subset of $W_\|$. In particular for each subset in $W_<$ of the form $u_{s\Downarrow} \cup v_{t\Downarrow} \cup x_\Downarrow$, for $j' < s < i$ and $l' < t < k$, we have the corresponding subset $u_{s\Downarrow} \cup v_{t\Downarrow} \cup x_\Downarrow$ in $W_\|$. Similarly, for each subset in $W_<$ of the form $u_{s\Downarrow} \cup v_{t\Downarrow} \cup x'_\Downarrow$, we have the corresponding subset $u_{s\Downarrow} \cup v_{t\Downarrow} \cup x_\Downarrow \cup x'_\Downarrow$ in $W_\|$. But then in addition to these, we have the $m \times n$ subsets of the form $u_{s\Downarrow} \cup v_{t\Downarrow} \cup x'_\Downarrow$ in $W_\|$ (each of which *do not* contain the element x), and there are no corresponding subsets to these in $W_<$.

Therefore, since $W_<$ and $W_\|$ contain different numbers of subsets, they cannot be compared with one another. But, as before, it is possible to add in some extra elements to $W_<$ in various ways, producing an expanded version $W^*_<$ of $W_<$ which does have the same number of subsets as does $W_\|$. For example, we could lengthen the chain U in $W_<$ by attaching $m \times n$ new elements $\{u_{p+1}, \ldots, u_{p+m\times n}\}$ with $u_p < u_{p+1}$, and then $u_i < u_{i+1}$, for all i between 1 and $p + m \times n$. To complete the picture, we assume that also $v_q < u_{p+1}$.

There are many other possibilities for adding new elements into $W_<$. For example we could add elements to the other chain V, to both, or midway along the chains at various positions, adjusting the number of new elements in each case so that the total number of subsets remains constant. Therefore we conclude that it is very probable that $x < x'$ if the index elements of x are all less than or equal to the corresponding index elements of x', becoming overwhelmingly probable when the indexing chains are long.

So let us assume that we have two distinct chains, U and V, and also many elements in a region which are indexed by these chains such that for any two of these elements, the indexing is not identical, and furthermore, given two such elements a and b, we have $a < b$ if and only if the indexing elements for a are all less than or equal to the corresponding indexing elements for b. Then we conclude that the set of these elements, considered as a partially ordered set, is 4-dimensional.

Recall the definition of dimension within the theory of partially ordered sets. Each partial order for a given set can be expanded by adding in further ordering relations to obtain a totally ordered set which contains the original partial order. A *realizer* of the partial ordering is a collection of total orders, each of which contains the original partial order, such that the original partial order is the intersection of all the ordering relations in the realizer. A partially ordered set has the dimension n if there is a realizer consisting of n totally ordered sets, where n is the smallest such number.

Applying this to our situation with the two chains U and V, we can find a realizer consisting of just 4 totally ordered sets. They are obtained by using the ordering of each of the four indexing elements. For example, given that $a\|b$, the first total ordering involves adding in the relation $a < b$ if the upper indexing element along U of a is less than or equal to the upper indexing element of b along U. In this way we obtain the first of our total orderings. The other three are obtained similarly.

When thinking about this argument, it might be objected that the same ideas could be applied to the situation with just one single indexing chain, thus seemingly leading to the conclusion that we would have 2 dimensions rather than 4. That is to say, let the single chain

$$U = \{u_1, \ldots, u_p : u_i < u_{i+1}, \forall 1 \leq i < p\}$$

be given, together with two elements x and x' not in the chain, but such that the chain has elements which are greater than both x and x' and also elements less than both x and x'. As before, we take the indexing elements for x to be u_i and u_j, and for x' to be $u_{i'}$ and $u_{j'}$. Let $W_\|$ be the configuration with $x\|x'$ and $W_<$ with $x < x'$.

Assuming there are m elements along the chain U between $u_{i'}$ and u_j, then there are m more subsets in $W_\|$ than there are in $W_<$. Following the argument as before, we must add in further elements to $W_<$ in order to be able to compare the two configurations with one another. There are a number of different ways to add in new elements, always preserving the total number of subsets, and these different ways give different pairs of indexing elements for x and x', all of which are equally probable. But since we have $x < x'$, all of these sets must preserve the condition that the indexing elements for x are less than (or equal to) the indexing elements for x'. On the other hand, for $W_\|$ we now have fewer elements than $W_<$ in the chain U, but more freedom to choose the indexing elements. All possibilities are open *except* the case where *both* indexing elements for x are less than *both* indexing elements for x' (for that would imply that we must have $x < x'$), or conversely, if both indexing elements for x' are less than both indexing elements for x. Taken together, the number of possible configurations with $x < x'$ is not greater than the number with $x \| x'$, and so the argument fails.

We see then that the presence of the second indexing chain, adding in so many further possible configurations in the case $x < x'$, is essential for our argument. Having three or more indexing chains adds nothing, since they will only confirm the correlation between the ordering of the elements and that of their indexing elements. After all, the dimension is given by the least possible number of total orderings in a realizer.

5 Conclusion

From the few considerations dealt with here, it is obvious that our very natural way of defining probabilities in finite partially ordered sets will lead to structures which depart strongly from what might at first be expected. Rather than having a chaos of unordered sets, we see that chain-like structures which might interact with one another in orderly ways are probable.