A Class of Partially Ordered Sets: IV
(Some A Priori Probability Considerations)

by Geoffrey Hemion

1 Introduction

We use the definitions of the preceding paper (A class of partially ordered sets: III). Thus for our purposes, a typical poset X will be considered to be an element of the class $\mathfrak{W}$, which was defined there. That is to say, X is discrete, upwardly separating, confluent below, extensional, such that all elements are interior.

Since such posets must be infinite, it is not possible to deal with the question of probabilities in these posets by simply counting all the elements in a given poset. Instead, it is necessary to find a sensible method of defining finite regions which can be compared with one another. The question then reduces to the problem of identifying which structures, or configurations, are most probable in such a finite region.

To a large extent, the assumptions we make will be guided by intuition. The goal is to find a discrete mathematics which could be used as a model for physics.

2 Localized positions

In the previous paper, the idea of a position in a poset was defined. A position is a pair (U, V) of non-empty subsets of the given poset X, such that $U \leq V$ (that is, $u \leq v$ for all $u \in U$ and $v \in V$), and which is *maximal*, in the sense that U is the largest possible subset which is beneath V, and V is the largest possible subset which is above U. Any element $x \in X$ provides us with an *elementary position*, namely $(x_{\Downarrow}, x_{\Uparrow})$, where $x_{\Downarrow} = \{y \in X : y \leq x\}$, and $x_{\Uparrow} = \{z \in X : z \geq x\}$.

Obviously, an elementary position is "localized" in the sense that it can be identified with a specific element of X. But what is the situation with respect to non-elementary positions?

To see what can happen, let us take Euclidean 3-space, $\mathbb{R}^3$, and we can imagine it being made discrete by replacing the given Euclidean structure with some uniformly dense, but discrete, network of points.

$\mathbb{R}^3$ can also be thought of as a partially ordered set. Given $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ then the ordering is given by saying that $x \leq y$ if and only if both $x_1 \leq y_1$ and $(y_2 - x_2)^2 + (y_3 - x_3)^2 \leq (y_1 - x_1)^2$. With this ordering, the elementary position $(x_\Downarrow, x_\Uparrow)$ is a double cone, with the upper cone meeting the lower cone in the point at the apex, which is the point $x \in \mathbb{R}^3$. Thus — given the causality ordering of physical space — it would seem to be a natural idea to associate positions with points of space.

Unfortunately though, most positions in $\mathbb{R}^3$ are not elementary positions. For example, let $l = \{(0, s, 0) \in \mathbb{R}^3 : |s| \leq 1\}$. Then take

$$U = \{y \in \mathbb{R}^3 : \exists x \in l, y \leq x\} \quad \text{and} \quad V = \{z \in \mathbb{R}^3 : z \geq u, \forall u \in U\}.$$

The pair (U, V) is a position in $\mathbb{R}^3$, yet it cannot be identified sensibly with any single point of $\mathbb{R}^3$. The problem illustrated in this example is that such generalized positions jump over a region of empty space between the upper and lower sets so that they can no longer be localized at the point where those two sets meet.

One way of describing this situation is to think of chains of related elements. A *chain* is a totally ordered subset of a given poset. A chain is *maximal* if it is not properly contained within another chain. In $\mathbb{R}^3$, the maximal chains are continuous lines (world-lines), passing upwards through the poset. Given an element $x \in \mathbb{R}^3$, then every maximal chain which contains x is completely contained within the elementary position $(x_\Downarrow, x_\Uparrow)$. This leads to the following definition, which is also applicable to discrete posets.

Definition 1. *Let (U, V) be a position in the poset X. The position will be called* localized *if there exists a maximal chain $C \subset X$ which is contained within the position. That is $C \subset U \cup V$.*

In case the position (U, V) is not elementary, then it is localized if there exist two elements $u \in U$ and $v \in V$, such that $u_\uparrow \cap v_\downarrow = \emptyset$.

(Here $u_\uparrow = \{y \in X : y > u\}$ and $v_\downarrow = \{z \in X : z < x\}$.)

Returning to the class $\mathfrak{W}$ of discrete posets and taking any pair of related elements $a < b$ in $X \in \mathfrak{W}$, we have that the subset $b_\downarrow \setminus a_\downarrow$ must be finite, while $a_\uparrow \setminus b_\uparrow$ must be infinite. As was shown in the previous paper, this implies that there can be at most finitely many positions between a and b. Therefore, within the class $\mathfrak{W}$, let us say that the volume of "space" between a and b is the number of localized positions between them.

From now on, only localized positions will be considered, and thus the word 'position' will always refer to a *localized* position. Furthermore, only posets $X \in \mathfrak{W}$ in our particular class of posets will be considered.

3 Why the elements form chains

Let $a < b$ be two related elements in the poset X. We assume that they are sufficiently separated that the volume of space between them is large; that is, there are N (localized) positions between a and b, where N is some large, yet fixed, number. It was shown in the previous paper that all of these positions are uniquely associated with elements of $b_\Downarrow \setminus a_\downarrow$. Therefore the fact that there are precisely N positions between a and b is related to the number and configuration of the elements of X in $b_\Downarrow \setminus a_\downarrow$.

We are concerned with identifying probable patterns, or configurations, within X (or indeed, within any arbitrarily chosen element of $\mathfrak{W}$). For this, and to allow a sensible comparison between similar things, the obvious procedure would be to examine the various possible configurations of finitely many elements which might be placed into $b_\Downarrow \setminus a_\downarrow$, such that we always have again N positions between a and b. Thus we are considering a kind of variational analysis, where the original poset X is varied to produce possible new posets X' which only differ from X at most in the subset $b_\downarrow \setminus a_\Downarrow$ (in particular, the elements a and b are not varied).

Now take two distinct elements x and y in $b_\downarrow \setminus a_\Downarrow$. The question is, with how many positions in $a_\Uparrow \cap b_\Downarrow$ are they associated? There are two different cases to consider, namely

1. they are related to one another — that is, either $x < y$ or $y < x$, written $x \perp y$, or else
2. they are unrelated, written $x \| y$.

But if $x \perp y$, then any position in $a_\Uparrow \cap b_\Downarrow$ (that is, a position *between* a and b, where a is an element of the lower set of the position and b is in the upper set) can be at most associated with one of the elements, x or y.

On the other hand, if $x \| y$ then a position in $a_{\Uparrow} \cap b_{\Downarrow}$ can be associated with either the element x or the element y singly, or it can also be associated with *both* x and y. Therefore, in the first case, there are only two different ways a position can be associated with x and/or y, while in the second case there are three different ways.

Given this, then it seems reasonable to say that a configuration of elements in $b_{\downarrow} \setminus a_{\Downarrow}$ having many pairs of unrelated elements would produce more positions in the space $a_{\Uparrow} \cap b_{\Downarrow}$ than a configuration with the same number of elements, yet where most of them form chains of related elements. Or put another way, if a variation is only allowed with a fixed number of positions between a and b, then a variation containing chains of elements in $b_{\downarrow} \setminus a_{\Downarrow}$ will have more elements than a variation which has few chains. But if we have more elements, then there are more possible ways to place them into $b_{\downarrow} \setminus a_{\Downarrow}$ in comparison with configurations with only few elements. The conclusion is that configurations with as few unrelated pairs as possible — that is, configurations where the elements form chains — are most probable.

Of course, taking this idea to an extreme, we arrive at a totally ordered set which has no pairs of unrelated elements at all. Yet a totally ordered set cannot be an element of our class of sets $\mathfrak{W}$. Therefore, given some poset $X \in \mathfrak{W}$, the question arises as to which configurations of pairs of unrelated elements are more, or less probable.

So let x and y be two elements of $X \in \mathfrak{W}$, with $x \| y$. Our argument that unrelated pairs are not probable is concerned with the possibility that both elements of a pair such as x and y may be associated with a single *localized* position (U, V). For this, let us take two elements $u < v$ in U, with $u_{\uparrow} \cap v_{\downarrow} = \emptyset$. Our argument is concerned with the set of all possible localized positions between such a pair of elements as u and v.

We consider two alternative situations. Namely, the two elements x and y of the unrelated pair are "close together", or else they are "far apart". But how should we define the "distance" between two unrelated elements? One possibility is to define it to be the number of elements in the set $(x_{\downarrow} \setminus y_{\downarrow}) \cup (y_{\downarrow} \setminus x_{\downarrow})$.

Can it be that there are many localized positions between u and v which are associated with both of the elements x and y together? Since u and v are adjacent (that is, $u_{\uparrow} \cap v_{\downarrow} = \emptyset$), we expect that there are few elements in $v_{\downarrow} \setminus u_{\downarrow}$. However, if both x and y are together associated with a single position between u and v, then they must both be elements of $v_{\downarrow} \setminus u_{\downarrow}$.

Let us say that we know that the element x is contained in $v_{\downarrow} \setminus u_{\downarrow}$. What is the probability that also y is contained in $v_\downarrow \setminus u_\downarrow$? If the set $y_\downarrow \setminus x_\downarrow$ is large — thus x and y are far apart — then there are many pairs like u, v which contain not only x, but also various elements of $y_\downarrow \setminus x_\downarrow$ in the set $v_\downarrow \setminus u_\downarrow$. Yet remember that $v_\downarrow \setminus u_\downarrow$ is itself small. Therefore the probability that one particular element of the large set $y_\downarrow \setminus x_\downarrow$, or indeed of y itself, also being contained in $v_\downarrow \setminus u_\downarrow$ is relatively small. On the other hand, if $y_\downarrow \setminus x_\downarrow$ is small, then there are fewer elements of $y_\downarrow \setminus x_\downarrow$ available to fill up the space $v_\downarrow \setminus u_\downarrow$. Thus it is more probable that $y \in v_\downarrow \setminus u_\downarrow$.

To summarize: Given a poset $X \in \mathfrak{W}$, and a pair of unrelated elements $x \| y$ in X, such that x and y are far apart — that is, if the set $(x_\downarrow \setminus y_\downarrow) \cup (y_\downarrow \setminus x_\downarrow)$ is large — then a variation of X which introduces a new ordering relation between x and y would not be expected to reduce the number of localized positions to the extent that would be the case if x and y are close together. Or in other words, if x and y are close together, then it is probable that they are related to one another. On the other hand, if they are far apart, then it is more probable that they are unrelated. A collection of elements which are close together tends to form chains; a widely spaced collection does not have such a strong tendency — unrelated elements tend to remain unrelated.

4 How the positions of elements are determined by other elements

Let $x \in X$ be some element in a poset $X \in \mathfrak{W}$. Then, of course, the pair $(x_\Downarrow, x_\Uparrow)$ is an elementary position in X. But now let us consider the pair $(x_\downarrow, x_\uparrow)$ in $X \setminus \{x\}$. If x is an essential element of X, then $(x_\downarrow, x_\uparrow)$ is not a position in the poset $X \setminus \{x\}$.[1] Still, $x_\downarrow$ is the lower set of a position in X, namely $(x_\downarrow, V)$, where $x_\Uparrow \subset V$ and $V \setminus x_\Uparrow \neq \emptyset$. Let us call this the position *directly beneath* the element x.

Thus, given the position $(x_\downarrow, V)$ directly beneath x, we see that all elements $v \in V \setminus x_\Uparrow$ are such that $x \| v$. On the other hand, since $v > x_\downarrow$, if v is a lowest element of $V \setminus x_\Uparrow$, a variation of X could be performed, adding in the single new relation $x < v$. According to our previous considerations, this will be probable if v is near to x. Therefore, given that the configuration around x is a probable one, we must conclude that all the elements of $V \setminus x_\Uparrow$

[1] That is, the element x is associated with itself. On the other hand if $(x_\downarrow, x_\uparrow)$ is a position in $X \setminus \{x\}$, then we say that x is an *inessential element*.

are far away from x. This means that locally — near to x — the pair $(x_\downarrow, x_\uparrow)$ does correspond with the position $(x_\downarrow, V)$ in $X \setminus \{x\}$.

Concentrating on the situation near to x, let us say that the element $a \in x_\downarrow$ is associated with the position directly beneath x. That means that there must be some element $b \not> a$, yet with $b > x_\downarrow \setminus \{a\}$. If b is nearer to x than a, then we can perform a variation, removing the relation $x > a$ (so that in the varied poset, we have $x \| a$), and adding in the new relation $x < b$. The net result is to have exchanged the close unrelated pair $x \| b$ for the more distant unrelated pair $a \| x$. Thus this variation leads to a more probable poset, and so such a configuration near to x is probable.

In a similar way, it might be the case that there is an element $c \| x$ which is such that all elements of $x_\uparrow \setminus c_\uparrow$ are further from x than is c. In this case, a variation adding in the new relation $c < x$ and removing the relations of x to all elements of $x_\uparrow \setminus c_\uparrow$, would also result in a more probable poset. In both cases we see that it is probable that (as far as is possible without changing their mutual relationships) the elements near to x are related to x.

We can also consider positions *above* the essential element x. Let (U, V) be a position which is greater than x (so that $x_\Downarrow$ is a proper subset of U), such that there is no other position between (U, V) and x. That is, (U, V) is a position *directly above* x. In contrast to the *single* position which is directly beneath x, there may be more than one position directly above x (assuming of course that they are non-elementary).

What possibilities are there for a non-elementary position (U, V) directly above x? Remembering that all positions are only associated with elements beneath the position, we see that we must have $U = x_\Downarrow \cup \{a_1, \ldots, a_n\}$, for some finite number of elements a_i, $i = 1, \ldots, n$, with $a_i \| x$, and then $V = U_\Uparrow$.

The simplest idea would be to simply choose some single element $a \in X$ with $a \| x$ and $a_\downarrow \subset x_\downarrow$. This would give us $U = x_\Downarrow \cup \{a\}$ and $V \subset x_\uparrow$. In general we can expect to have many such elements as a, and so we would have many different positions directly above x.

But is it probable that there are, in fact, many different positions directly above x? Let us examine a position (U, V) whose lower set is of the form $x_\Downarrow \cup \{a\}$. Thus $U = x_\Downarrow \cup \{a\}$ and all elements of V are above both x and a. However, for all further elements $b \| x$ with $b \neq a$, there must be some element $z \in V$ with $z \not> b$.

Given such a b near to x in the sense that $b_\downarrow \subset x_\downarrow$, let us take a lowest $z \in V$ with $z \not> b$. If z is not far away from x, our argument shows that a variation which introduces the new relation $z > b$ is probable. This would bring with it also the new relations $y > b$, for all the elements $y \in z_{\uparrow}$. The same could be said for other elements $z' \in V$ with $z' \| z$ and $z' \not> b$. Thus we would have to add in the element b to the lower set of our position directly above x.

So the conclusion we draw is that it is most probable that there are relatively few different positions directly above x, and given such a position (U, V), then the set $U \setminus x_{\Downarrow}$ contains a relatively large (but of course only finite) number of elements.

5 Chains

If it is more probable that nearby elements are related, rather than being unrelated, then it follows that in a typical poset $X \in \mathfrak{W}$, the elements will tend to form discrete chains, the adjacent elements of which are close together. Let $\mathcal{C}$ be typical chain. Perhaps it is infinitely long, or perhaps it is only finite. Let $x_1 < x_2 < \cdots < x_n$ be some finite segment of adjacent elements along $\mathcal{C}$. Now take some other chain $\mathcal{C}'$, disjoint from $\mathcal{C}$, which is sufficiently long that it contains elements less than x_1 and also elements greater than x_n. Given some particular element x_i of $\mathcal{C}$, then if $\mathcal{C}$ and $\mathcal{C}'$ are far apart, we expect to have many elements of $\mathcal{C}'$ being unrelated to the element x_i. On the other hand, if $\mathcal{C}$ and $\mathcal{C}'$ are close together, then there will be fewer elements of $\mathcal{C}'$ which are unrelated to x_i. Does this mean that it is more probable that $\mathcal{C}$ is close to $\mathcal{C}'$?

In fact, our previous argument cannot be applied to chains. Recall that if (U, V) is some position, then if two given elements a and b are related to one another, we can only have the position being associated with at most one of the elements, *a or b*. On the other hand, if a and b are unrelated, then the position could — in addition — be associated with *both* a and b together. So the conclusion was that a configuration with a being related to b would be more probable.

But now take the two chains $\mathcal{C}$ and $\mathcal{C}'$, and again consider some position (U, V) in X. Assuming that the chains are long enough to contain both elements in U, and also elements not in U, then the position can be associated with at most a single element from each chain — either one element from one of the chains, or two elements, namely one element from the chain $\mathcal{C}$ and another element from the chain $\mathcal{C}'$. This is true regardless of whether or not the two chains are close together; in either case, just a single element of each chain is available to be associated with the position.

On the other hand, an argument can be made that a kind of "generalized" chain might be probable. That is to say, given two distinct chains $\mathcal{C}$ and $\mathcal{C}'$, they might be so close together that each element of each chain is only unrelated to a single element of the other chain. Thus, if $x_1 < \cdots < x_n$ is a segment of $\mathcal{C}$ and $x'_1 < \cdots < x'_n$ the corresponding segment of $\mathcal{C}'$, then we have $x_i \| x'_i$ for each i, yet $x_i < x'_j$ and $x'_i < x_j$ if $i < j$. Let us now imagine that $\mathcal{C}$ is near to $\mathcal{C}'$, in the sense that both $x_{i\downarrow} \setminus x'_{i\downarrow}$ and $x'_{i+1\downarrow} \setminus x_{i\downarrow}$ have few elements, for each i. In this case it is unlikely that a randomly chosen position (U, V) in X will be associated with both an element of $\mathcal{C}$ and also an element of $\mathcal{C}'$. Instead, just a single element from the union of the two chains $\mathcal{C} \cup \mathcal{C}'$ would be more likely.

More generally, this argument shows that generalized chains of the form $\{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$, with the relations generated by $x_i < x_{i+m}$, for all i and for some fixed $m > 1$, might also be probable.

If a poset X consists mainly of chains whose adjacent elements are close together then there will be only few localized non-elementary positions, and each will be closely linked to a single (perhaps generalized) chain. The number of these non-elementary positions between adjacent elements of the chain gives the measure of a kind of "length" of the chain, thus providing a geometry for X. In a previous paper we have discussed the way the number of positions is related to the "density" of elements near to the chain. But going beyond this, there is an additional effect which might be considered. The non-elementary positions between elements of a chain might — to some small extent — be "spread-out", as in the example which was discussed in section 2. If, however, there are many elements of X near to, but not contained in, a given chain, then their presence (in comparison with the situation where they are not present) will prevent some of these spread-out positions from occurring.