On Finite Separating Posets

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Our investigations have been concerned with infinite partially ordered sets which contain neither maximal nor minimal elements. The question of probabilities within such posets plays a large role. Unfortunately though, when thinking about constructing models in order to perform possible computer simulations, we are confronted with the obvious fact that computers are finite. Thus, for such purposes, we must look for an altered framework of finite posets which nevertheless retains some of the properties of the infinite posets which are of interest. In the present paper therefore, all posets will be considered to be finite.

1 Definitions

Let $(X, \leq)$ be a (finite) poset. Let $x \in X$. We write $x_\downarrow = \{y \in X : y < x\}$ and $x_\Downarrow = \{y \in X : y \leq x\}$. The element x is *maximal* if $x_\uparrow = \emptyset$, and it is *minimal* if $x_\downarrow = \emptyset$. If it is neither maximal nor minimal, then it is an *interior element*.

The poset $(X, \leq)$ is *separating* if:

- For any two interior elements x and y with $x \neq y$, we have both $x_\downarrow \neq y_\downarrow$, and also $x_\downarrow \neq y_\Downarrow$.
- For any two elements u and v which are not maximal and which are such that $u \not\leq v$, we have $v_\uparrow \setminus u_\uparrow \neq \emptyset$.

Given the set X with two different partial orderings $\leq_1$, $\leq_2$, we will say that $\leq_2$ is an *extension* of $\leq_1$ if $\leq_1 \subset \leq_2$. (Here, partial orderings are considered to be subsets of the Cartesian product $X \times X$.) We will say that a separating poset $(X, \leq)$ which is such that there is no separating extension involving pairs of non-maximal elements is a *maximally extended* separating poset.

2 Elementary Properties

Let $(X, \leq)$ be some arbitrary finite poset. A *chain* in X is a totally ordered subset (considered with the ordering of X); an *antichain* is a subset which is such that for any pair of elements, no ordering relation exists. Clearly both the set of all maximal elements and the set of all minimal elements are antichains. The *height* of X is the length of the longest possible chain. The *width* is the greatest number of elements which can be found in an antichain of X.

Theorem 1. *Let $y \not< x$ be two non-maximal elements in a separating poset. Then there exists a maximal element in the set difference $x_\uparrow \setminus y_\uparrow$.*

Proof. There must be an element $z \in x_\uparrow \setminus y_\uparrow$, which of course is greater than x. If z is not maximal, then there must be a further element $z' \in z_\uparrow \setminus y_\uparrow$, and we again have $z' \in x_\uparrow \setminus y_\uparrow$. Now $x < z < z'$. Continuing this process, we arrive at a maximal element. □

Theorem 2. *The height of a separating poset is no greater than the number of its maximal elements.*

Proof. Let $x_1 < x_2 < \cdots < x_n$ be a longest possible chain in X. Then each of $(x_{i+1})_\uparrow \setminus (x_i)_\uparrow$ contains a maximal element, and they must be all different. □

Theorem 3. *A separating poset with only one minimal element has no interior elements. If it has two minimal elements then there can be at most one interior element.*

Proof.

1. If the separating poset X has just one minimal element a, assume there also exists an interior element. Let x be a minimal interior element. Then we would have $x_\downarrow = a_\Downarrow$, which is impossible.
2. If two minimal elements a and b, then if x is an interior element it must have $x_\downarrow = \{a, b\}$. Then no further interior elements, either above or unrelated to x are possible.

□

Theorem 4. *Let $(X, \leq)$ be any finite poset satisfying the property: For any two interior elements x and y with $x \neq y$ we have both $x_\downarrow \neq y_\downarrow$, and also $x_\downarrow \neq y_\Downarrow$. Then there exists a separating poset $(X', \leq')$ containing X as a subposet, such that $X' \setminus X$ consists only of the maximal elements of X'.*

Proof. For every element $x \in X$, let $\bar{x}$ be a new element not in X with only the relations given by $\bar{x} > x_\Downarrow$. Thus for each x, the new element $\bar{x}$ is a maximal element in X'. Given any two elements u and v in X' which are not maximal and which are such that $u \not\leq v$, we have $\bar{v} \in v_\uparrow \setminus u_\uparrow \neq \emptyset$. Also if x is an interior element of X', we cannot have $x_\downarrow = y_\downarrow$ or $x_\downarrow = y_\Downarrow$ for any of the maximal elements of X'. □

Note that the proof of this theorem involves an overabundance of additional maximal elements. Clearly the result is also true if we only require that for each pair of elements x, y, with $x \not< y$, we have a maximal element v in X' with $v > y$ and $v \not> x$.

Theorem 5. *A maximally extended separating poset with only three minimal elements is such that the width of the interior elements is at most three.*

Proof. Assume to the contrary that there exists a maximally extended separating poset with only three minimal elements, yet which contains an antichain of interior elements with more than three elements. Let $(X, \leq)$ be a smallest such a poset in the sense that any such poset with fewer interior elements satisfies the conditions of the theorem. Theorem 4 shows that we may assume that the antichain A, consisting of the maximal interior elements of X, contains more than three elements, while all other antichains of interior elements contain no more than three elements. Let A' be the antichain directly beneath A. That is, if A is removed from X, then A' will be the new set of maximal interior elements. According to our hypotheses, A' consists of at most three elements. Thus one element of A, call it $a \in A$, must have $A' \subset a_{\downarrow}$. Also any other element $b \in A$ must have $b_{\downarrow} \cap A'$ consisting of two elements. But then X may be extended, adding in the new relations $b < a_{\Uparrow}$. Restricting our attention to the non-maximal elements and using Theorem 4, we obtain a contradiction. □

3 Separating posets which can be extended

Begin by observing that according to theorem 1, the property that a poset is separating means that there must be a maximal element in $v_{\uparrow} \setminus u_{\uparrow}$.

Theorem 6. *Let $(X, \leq)$ be a separating poset. Assume that there exists an interior element $x \in X$ such that*

1. *there exists a maximal element $a \not> x$ such that $x_{\downarrow} \subset a_{\downarrow}$, and*
2. *there exists a further maximal element $b \neq a$ with $b \not> x$, and*
3. *there exists no pair of interior elements y and z with $y_{\downarrow} = z_{\downarrow} \cup \{x\}$ or $y_{\downarrow} = z_{\Downarrow} \cup \{x\}$, where all the maximal elements of $z_{\uparrow}$ are contained in $x_{\uparrow} \cup \{a\}$.*

Then $(X, \leq)$ is not a maximally separating poset.

Proof. Let x be an *uppermost* such interior element. That is, if y is also an interior element of X satisfying the conditions of the theorem, then the number of elements in $y_{\downarrow}$ is not greater than the number of elements in $x_{\downarrow}$.

Then take the following extension $\leq_1$ of $\leq$. Namely in addition to the relations in $\leq$, we have $\leq_1$ containing the new relation $x < a$, and furthermore, and for all $v \not< x$ such that $a > v$ and all other maximal elements which are greater than v are also greater than x, we have the new relation $x < v$ in $\leq_1$.

In order to prove the theorem, we must show that $(X, \leq_1)$ is again a separating poset.

$(X, \leq_1)$ is a poset

In order to show this, it is necessary to show that

1. if $u < x$ then we have both $u < a$ and also $u < v$, for all v satisfying our condition, and
2. if $w > v$, for some v satisfying our condition, then also $x < w$.

But since we have assumed that $x_{\downarrow} \subset a_{\downarrow}$, we have $u < x$ implies $u < a$. On the other hand, if $u < x$ but $u \not< v$, then since the original poset $(X, \leq)$ is separating, we must have some $b \in v_{\uparrow} \setminus u_{\uparrow}$. We may assume that this b is a maximal element. However, since $u \in x_{\downarrow}$, we have $u < a$. Thus b is not a, and also $b \not> x$, since $u < x$. This contradicts the condition that the only maximal element of $v_\uparrow$ not in $x_\uparrow$ is a.

Now assume that $w > v$. Then we must have $w_\uparrow \subset v_\uparrow$ in the original ordering $\leq$. But then, apart from a, all the maximal elements in $w_\uparrow$ are contained in $x_\uparrow$. Thus in the ordering $\leq_1$, we must have $x < w$.

$(X, \leq_1)$ **is separating**

Here it is necessary to show that with respect to the ordering $\leq_1$ we have that

1. if $y \neq z$ and y are interior elements, then $y_\downarrow \neq z_\downarrow$ and $y_\downarrow \neq z_\Downarrow$, and also
2. if $c \ngeq d$ are two elements, neither of which is maximal, then $c_\uparrow \setminus d_\uparrow \neq \emptyset$.

To deal with the first point, let y be an interior element, and let us assume that in the ordering $\leq_1$ we have $y_\downarrow = z_\downarrow$. Since $y_\downarrow \neq z_\downarrow$ in the ordering $\leq$, and since the only element which is the lower element in a new relation in $\leq_1$ which wasn't in $\leq$ is x, it must be that say $y_\downarrow = z_\downarrow \cup \{x\}$ in the ordering $\leq$. Since $z \notin y_\downarrow$, we do not have $y > z$. Therefore let e be a maximal element in $y_\uparrow \setminus z_\uparrow$ (again, considered in the original ordering $\leq$). We then have $z_\downarrow \subset e_\downarrow$. Since in the change from $\leq$ to $\leq_1$ we have the new relation $x < z$, it follows that $a > z$, yet no other maximal elements not in $x_\uparrow$ are greater than z. Thus, in particular, there exists another maximal element $b \neq a$ which is neither greater than x nor greater than z. However $e > x$, thus $e \neq b$. It follows that there are at least two maximal elements not greater than z. Thus the pair of elements z and e satisfy our conditions, and furthermore, $z_\downarrow$ is larger than $x_\downarrow$, since $x_\downarrow \subset z_\downarrow$ in the original ordering $\leq$, and furthermore $x_\downarrow \neq z_\downarrow$ since the original poset was separating. This contradicts the assumption that x was an uppermost candidate for making our extension, and therefore the assumption that in the extension $\leq_1$ we have $y_\downarrow = z_\downarrow$ must be false.

Could it be that $y_\downarrow = z_\Downarrow$? We cannot have $z = x$, for otherwise in the original poset we would have $y_\downarrow = x_\downarrow$. Therefore it must be that in the original poset we have $y_\downarrow = z_\Downarrow \cup \{x\}$, with $x \not< z$, yet in the altered poset, $x < z$. But this case has been excluded.

Finally, for the second point, let $c \ngeq d$ be two elements, neither of which is maximal. If neither c nor d is x, then since their upper sets are the same, both in $\leq$ and in $\leq_1$, we must have $c_\uparrow \setminus d_\uparrow \neq \emptyset$. Also if $c = x$ then $x_\uparrow \setminus d_\uparrow \neq \emptyset$ in $\leq_1$, since in the extension, $x_\uparrow$ contains more elements than in the ordering $\leq$, and $d_\uparrow$ remains unchanged. Thus the only interesting case is that $d = x$. If a is the only maximal element in the original ordering $\leq$ which is greater than c, yet not greater than x, then in the extension $\leq_1$ we would have $c > x$, which we have assumed not to be true. Therefore there exists some other maximal element $b \neq c$ in $c_\uparrow \setminus x_\uparrow$ in the original ordering $\leq$. And therefore in the extension $\leq_1$, we also must have $b \in c_\uparrow \setminus x_\uparrow$. □

This method of proof defines an algorithm for extending a separating poset such that after each extension, the poset remains separating. Furthermore the number of minimal and maximal elements remains unchanged. In particular, the height of any chain is limited by the number of maximal elements. Thus if the poset has many more interior elements than maximal elements, the width of the interior elements in any extension will remain large.

On the other hand, thinking in terms of the infinite posets which motivate this theory, there will always be a sufficient "density" of elements above a given region to ensure that the condition $x_\uparrow \setminus y_\uparrow \neq \emptyset$ holds for all $x \ngeq y$. Thus in this finite framework, and with regard to theorem 4, we might think that the maximal elements should always be present in sufficient numbers in order to ensure that the condition is satisfied in any process of extensions.

4 Essentially separating posets

Definition 1. *A finite poset will be called* essentially separating *if for any two interior elements x and y with $x \neq y$, we have both $x_\downarrow \neq y_\downarrow$, and also $x_\downarrow \neq y_\Downarrow$.*

This is slightly more restrictive than the axiom of extensionality in Zermelo-Fraenkel set theory. In the language of sets, we are also excluding the possibility that a set has only one element which is not the empty set.

As before, we consider extensions of given posets, but now we will say that an essentially separating poset $(X, \leq)$ is essentially maximally extended if there is no separating poset $(X_1, \leq_1)$ which extends $(X, \leq)$.

Theorem 7. *Let $(X, \leq)$ be an essentially separating poset. Assume that there exists an interior element $x \in X$ such that*

1. *there exists a further interior element $a \ngtr x$ such that $x_\downarrow \subset a_\downarrow$, and*
2. *there exists no pair of interior elements b and c with $b_\downarrow = c_\downarrow \cup \{x\}$ or $b_\downarrow = c_\Downarrow \cup \{x\}$, such that $c \geq a$.*

Then $(X, \leq)$ can be extended by including the extra relations given by $x < a_\Uparrow$, and the resulting poset is still essentially separating.

Proof. We denote by $\leq_1$ the extension of $\leq$ given by including the relations $x < a_\Uparrow$.

First of all, it is clear that since $x_\downarrow \subset a_\downarrow$, then it follows that $(X, \leq_1)$ is a poset. Is it essentially separating?

Let u and v be distinct elements. Can it be that $u_\downarrow = v_\downarrow$ in the ordering $\leq_1$, despite the fact that $u_\downarrow \neq v_\downarrow$ in the ordering $\leq$? This could only be the case if, say, $v_\downarrow \subset u_\downarrow$ and $u_\downarrow \setminus v_\downarrow = \{x\}$ in the ordering $\leq$, with $v \geq a$. However we have excluded this possibility in the statement of the theorem.

Could we have two elements u and v with $u_\downarrow = v_\Downarrow$ in the ordering $\leq_1$? We cannot have $v = x$ and $u = a$ since in the original ordering $\leq$, we do not have $x_\downarrow = a_\downarrow$. Otherwise, we must have $u_\downarrow = c_\Downarrow \cup \{x\}$ with $c \geq a$. But again, this possibility has been excluded. □

It follows that in a non-trivial, essentially maximally extended separating poset, all non-maximal elements x are such that there exists a pair u, v of further elements, such that either $u_\downarrow = v_\downarrow \cup \{x\}$ or $u_\downarrow = v_\Downarrow \cup \{x\}$. To see this, begin by observing that if x is a minimal element then $x_\downarrow = \emptyset$, so that for any non-minimal element $v \not> x$, we have $x_\downarrow \subset v_\downarrow$, and the assertion follows from the theorem. If x is not a minimal element, then in order to obtain a contradiction we assume that x is an interior element such that for all $v \not> x$, we do not have $x_\downarrow \subset v_\downarrow$. It may be further assumed that no elements of $x_\downarrow$ have this property. Let $v \not> x$ be chosen such that $x_\downarrow \setminus v_\downarrow$ has the fewest number of elements. Then take some $y \in x_\downarrow \setminus v_\downarrow$ which is minimal within that set. Then there must be a u with $u_\downarrow = v_\downarrow \cup \{y\}$ or $u_\downarrow = v_\Downarrow \cup \{y\}$. In either case we have $u \not> x$, and yet $x_\downarrow \setminus u_\downarrow$ has fewer elements than $x_\downarrow \setminus v_\downarrow$, giving a contradiction.

Therefore, for such posets we can imagine that they consist of generalized "chains" of the form $x_1 \to x_2 \to \cdots \to x_n$, such that for each $k > 2$ we have either $x_{k\downarrow} = x_{k-1\downarrow} \cup \{x_{k-2}\}$ or $x_{k\downarrow} = x_{k-1\Downarrow} \cup \{x_{k-2}\}$.

In fact, as the next theorem shows, essentially maximally extended separating posets consist of a very simple form of generalized chains.

Theorem 8. *Let $(X, \leq)$ be an essentially maximally extended separating poset and let X' be the set of interior elements of X. Then there is just one maximal element of X', and otherwise, all maximal antichains contain just two elements.*

Proof. Let a be a maximal element in X'. Assume that there exists some other maximal element b. We must have either $a_\downarrow \setminus b_\downarrow \neq \emptyset$ or $b_\downarrow \setminus a_\downarrow \neq \emptyset$. Assume that $b_\downarrow \setminus a_\downarrow \neq \emptyset$ and let c be a minimal element in that set. Can it be that there is a $d \in X'$ with $d_\downarrow = a_\downarrow \cup \{c\}$? If so, then X may be extended by taking the new relation $a < d$, contradicting the fact that X is essentially maximally extended. Otherwise take the new relation $c < a$, again giving a contradiction.

Therefore let a be the single the maximal element of X'. Let $X'' = X' \setminus \{a\}$. There must be at least two maximal elements in X''. But if there were more than two, the same argument as before would show that X was not essentially maximally extended. So let us say that b and c are the two maximal elements of X''.

Take $X''' = X'' \setminus \{b, c\}$. We may assume that say $b_\downarrow \setminus c_\downarrow \neq \emptyset$. If also $c_\downarrow \setminus b_\downarrow \neq \emptyset$ then let d be a minimal element of $c_\downarrow \setminus b_\downarrow$. But now X can be extended by taking the new relation $d < c$, giving again a contradiction. Therefore we may assume that $c_\downarrow \setminus b_\downarrow = \emptyset$. Similarly, we may assume that $b_\downarrow \setminus c_\downarrow$ consists of just one single element, call it d.

We now have the situation that the only antichain containing the element b is $\{b, c\}$. Also the only two antichains which contain c are $\{b, c\}$ and $\{c, d\}$. Continuing downwards through the poset, we see that in fact X'' can be described as the generalized chain $\{x_1, \ldots, x_n\}$, with $x_i < x_j$ when $i < j - 1$. □

5 Why take maximally extended posets?

The motivation for these ideas comes from considering probabilities in infinite posets. We have argued that when comparing finite subsets of different posets with one another, it is natural to use the idea of "positions" in posets. Given a poset $(X, \leq)$, a position is a pair of subsets (U, V) such that for all $u \in U$ and $v \in V$ we have $u \leq v$, and furthermore, the pair is as large as possible in the sense that if $u' \leq v$ for all $v \in V$ then $u' \in U$, and similarly if $v' \geq u$ for all $u \in U$, then $v' \in V$. We might call U the lower set, and V the upper set of the position. Within our present theory of finite posets, let us only consider positions which are such that both the lower and upper sets contain interior elements of the poset.

Probabilities would then be given by specifying two numbers, namely the number of interior elements n, and the number of positions p in the essentially separating posets which are to be considered. Given this, then all the posets satisfying these conditions are taken to be equally probable.

The idea of looking at extensions of separating posets follows from the observation that an extended poset generally has fewer positions. It follows that if p is small in comparison with n, then randomly chosen posets will often have to be extended in order to reduce the number of their positions to p. In this connection it is obvious that theorem 8 is of little use, since the single, given structure which it describes would play only a minor role in determining probabilities.