Stratification of triangulated categories

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Maurice Auslander: Coherent functors

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Coherent Functors *

By

Maurice Auslander

Let $\mathscr{C}$ be an abelian category and F a (covariant) functor from $\mathscr{C}$ to abelian groups. We say that F is a coherent functor if there exists an exact sequence $(X, \_) \to (Y, \_) \to F \to 0$ where (X, A) denotes the maps from X to A. The main purpose of this paper is to initiate a study of the full subcategory $\check{\mathscr{C}}$ of coherent functors and give some applications to the theory of complexes in abelian categories as well as to some more specialized questions concerning modules over rings.

The first two sections of the paper are devoted to questions of notation and some of the more elementary questions concerning the category $\check{\mathscr{C}}$. For instance, it is shown that if $0 \to F_1 \to F_2 \to F_3 \to F_4 \to 0$ is an exact sequence of functors with F_2 and F_3 coherent, then F_1 and F_4 are also coherent. It is also shown that $\check{\mathscr{C}}$ is closed under extensions. Thus if X is a complex in C, then the cohomology functors $H^i(X, \_)$ are in $\check{\mathscr{C}}$. Since $\check{\mathscr{C}}$ has enough projectives, it makes sense to talk about the global dimension of $\check{\mathscr{C}}$. It is shown that the gl. dim $\check{\mathscr{C}} = 0$ or 2 and that gl. dim $\check{\mathscr{C}} = 0$ if and only if the gl. dim $\mathscr{C} = 0$.

We of course have the usual right exact functor $u : \mathscr{C} \to (\check{\mathscr{C}})^0$ given by $u(A) = (A, \_)$. If $\mathscr{D}$ is an abelian category, then we have the induced functor $u. : \mathscr{H}om(\mathscr{C}, \mathscr{D}) \to \mathscr{H}om((\check{\mathscr{C}})^0, \mathscr{D})$. Section two ends by showing that the functor $u.$ always has an adjoint $u^{\cdot} : \mathscr{H}om((\check{\mathscr{C}})^0, \mathscr{D}) \to \mathscr{H}om(\mathscr{C}, \mathscr{D})$ having the following properties: a) if $F : \mathscr{C} \to \mathscr{D}$, then $(u^{\cdot} F) u = F$ and $u^{\cdot} F$ is left exact and $u^{\cdot} F$ is exact if F is right exact; b) if $\mathscr{C}$ has enough projectives and $F : \mathscr{C} \to \mathscr{D}$ is right exact, then $(L^i F)(A) = u^{\cdot} F(\mathrm{Ext}^i(A, \_))$, where $u^{\cdot} F$ is exact.

As seen above, the identity functor $I : \mathscr{C} \to \mathscr{C}$ can be factored through $(\check{\mathscr{C}})^0$ as $I = (u^{\cdot} I) u$ where $u^{\cdot} I : (\check{\mathscr{C}})^0 \to \mathscr{C}$ is exact. It is this functor $u^{\cdot} I$ which is studied in section three. Denoting by $\check{\mathscr{C}}_0$, the full subcategory of $\check{\mathscr{C}}$ such that $u^{\cdot} I$ sends the objects in $(\check{\mathscr{C}}_0)^0$ to zero, we have that $\check{\mathscr{C}}_0$ is a dense subcategory of $\mathscr{C}$ and that $\mathscr{C}$ is equivalent to $(\check{\mathscr{C}}/\check{\mathscr{C}}_0)^0$. Since

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Auslander's formula

Fix an abelian category C. A functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is coherent if it fits into an exact sequence

$$\mathrm{Hom}_{\mathsf{C}}(-,X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-,Y) \longrightarrow F \longrightarrow 0.$$

Let $\mathrm{mod}\,\mathsf{C}$ denote the (abelian) category of coherent functors.

Theorem (Auslander, 1965)

The Yoneda functor $\mathsf{C} \to \mathrm{mod}\,\mathsf{C}$ admits an exact left adjoint which induces an equivalence

$$\frac{\mathrm{mod}\,\mathsf{C}}{\mathrm{eff}\,\mathsf{C}} \xrightarrow{\sim} \mathsf{C}$$

(where $\mathrm{eff}\,\mathsf{C}$ denotes the full subcategory of effaceable functors).

A motivating problem: vanishing of Hom

Fix a triangulated category T with suspension $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$.

Problem

Given two objects X, Y, find invariants to decide when

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(X,\Sigma^n Y) = 0.$$

This talk provides:

- a survey on what is known (based on examples)
- some recent results (joint with D. Benson and S. Iyengar)
- open questions

Vanishing of Hom: a broader perspective

Given objects X, Y in a triangulated category T, the full subcategories

$$X^{\perp} := \{Y' \in \mathsf{T} \mid \mathrm{Hom}^*_{\mathsf{T}}(X, Y') = 0\}$$

$$^{\perp}Y := \{X' \in \mathsf{T} \mid \mathrm{Hom}^*_{\mathsf{T}}(X', Y) = 0\}$$

are thick, i.e. closed under suspensions, cones, direct summands.

Note: The thick subcategories of T form a complete lattice.

Problem

Describe the lattice of thick subcategories of T.

Classifying thick subcategories: the pioneers

Thick subcategories have been classified in the following cases:

- The stable homotopy category of finite spectra [Devinatz–Hopkins–Smith, 1988]
- The category of perfect complexes over a commutative noetherian ring [Hopkins, 1987] and [Neeman, 1992]
- The category of perfect complexes over a quasi-compact and quasi-separated scheme [Thomason, 1997]
- The stable module category of a finite group [Benson–Carlson–Rickard, 1997]

All these cases have in common:

- The triangulated category is essentially small.
- A monoidal structure plays a central role (thus providing a classification of all thick tensor ideals).

DEFINITION (NEEMAN, 1996)

A triangulated category T with set-indexed coproducts is compactly generated if there is a set of compact objects that generate T, where an object X is compact if $\mathrm{Hom}_{\mathsf{T}}(X, -)$ preserves coproducts.

Examples:

- The derived category $\mathsf{D}(\mathsf{Mod}\, A)$ for any ring A. The compact objects are (up to isomorphism) the perfect complexes.
- The stable module category $\mathsf{StMod}\, kG$ for any finite group G and field k. The compact objects are (up to isomorphism) the finite dimensional modules.

Localising and colocalising subcategories

Fix a compactly generated triangulated category T.

Note: T has set-indexed products (by Brown representability).

Definition

A triangulated subcategory $\mathsf{C} \subseteq \mathsf{T}$ is called

- localising if C is closed under taking all coproducts,
- colocalising if C is closed under taking all products.

Problem

Classify the localising and colocalising subcategories of T*. Do they form a set (or a proper class)?*

Vanishing of Hom: support and cosupport

Let R be a graded commutative noetherian ring and T an R-linear compactly generated triangulated category.

We assign to X in T

- the support $\operatorname{supp}_R X \subseteq \operatorname{Spec} R$, and
- the cosupport $\operatorname{cosupp}_R X \subseteq \operatorname{Spec} R$,

where $\operatorname{Spec} R =$ set of homogeneous prime ideals.

Theorem (Benson–Iyengar–K, 2012)

The following conditions on T are equivalent.

- T *is stratified by* R.
- *For all objects* X, Y *in* T *one has*

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0 \iff \operatorname{supp}_R X \cap \operatorname{cosupp}_R Y = \varnothing.$$

Stratified triangulated categories

Definition

An R-linear compactly generated triangulated category T is stratified by R if for each $\mathfrak{p} \in \operatorname{Spec} R$ the essential image of the local cohomoloy functor $\Gamma_{\mathfrak{p}} \colon \mathsf{T} \to \mathsf{T}$ is a minimal localising subcategory of T.

Examples:

- The derived category $\mathsf{D}(\operatorname{Mod} A)$ of a commutative noetherian ring A is stratified by A [Neeman, 1992].
- The stable module category $\operatorname{StMod} kG$ of a finite group is stratified by its cohomology ring $H^*(G, k)$ [Benson–Iyengar–K, 2011].

Support and cosupport

Fix an R-linear compactly generated triangulated category T. For an object X define

- $\operatorname{supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathfrak{p}}(X) \neq 0\}$
- $\operatorname{cosupp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid \Lambda^{\mathfrak{p}}(X) \neq 0\}$

where $\Lambda^{\mathfrak{p}}$ is the right adjoint of the local cohomology functor $\Gamma_{\mathfrak{p}}$.

Theorem (Benson–Iyengar–K, 2011)

Suppose that T is stratified by R. Then the assignment

$$\mathsf{T} \supseteq \mathsf{C} \longmapsto \operatorname{supp}_R \mathsf{C} := \bigcup_{X \in \mathsf{C}} \operatorname{supp}_R X \subseteq \operatorname{Spec} R$$

induces a bijection between

- *the collection of localising subcategories of T, and*
- *the collection of subsets of $\operatorname{supp}_R \mathsf{T}$.*

Costratification

There is an analogous theory of costratification for an R-linear compactly generated triangulated category T:

- Costratification implies the classification of colocalising subcategories.
- Costratification by R implies stratification by R (the converse is not known).
- When T is costratified, then the map $\mathsf{C} \mapsto \mathsf{C}^{\perp}$ gives a bijection between the localising and colocalising subcategories of T.

- The derived category $\mathsf{D}(\operatorname{Mod} A)$ of a commutative noetherian ring A is costratified by A [Neeman, 2009].
- The stable module category $\operatorname{StMod} kG$ of a finite group is costratified by its cohomology ring $H^*(G, k)$ [Benson–Iyengar–K, 2012].

Tensor triangular geometry

For an essentially small tensor triangulated category $(\mathsf{T}, \otimes, 1)$ Balmer introduces a space $\operatorname{Spc}\mathsf{T}$ and a map

$$\mathsf{T} \ni X \longmapsto \operatorname{supp} X \subseteq \operatorname{Spc}\mathsf{T}$$

providing a classification of all radical thick tensor ideals of T.

This amounts to a reformulation of Thomason's classification when $\mathsf{T} = \mathsf{D}^{\mathrm{perf}}(X)$ (category of perfect complexes) for a quasi-compact and quasi-separated scheme X, because $\operatorname{Spc}\mathsf{T}$ identifies with the Hochster dual of the underlying topological space of X.

Kock and Pitsch offer an elegant point-free approach.

Example: quiver representations

Fix a finite quiver $Q = (Q_0, Q_1)$ and a field k. Set

- $\operatorname{mod} kQ$ = category of finite dimensional representations of Q
- $W(Q) \subseteq \operatorname{Aut}(\mathbb{Z}Q_0)$ Weyl group corresponding to Q
- $\operatorname{NC}(Q) = \{x \in W(Q) \mid x \leq c\}$ set of non-crossing partitions (c the Coxeter element, $\leq$ the absolute order)

Theorem (K, 2012)

The map

$$\mathrm{D}^b(\operatorname{mod} kQ) \supseteq \mathsf{C} \longmapsto \operatorname{cox}(\mathsf{C}) \in \operatorname{NC}(Q)$$

induces a bijection between

- *the admissible thick subcategories of* $\mathrm{D}^b(\operatorname{mod} kQ)$*, and*
- *the non-crossing partitions of type* Q*.*

Quiver representations: vanishing of Hom

- A thick subcategory is admissible if the inclusion admits a left and a right adjoint.
- The proof uses that the admissible subcategories are precisely the ones generated by exceptional sequences.
- If Q is of Dynkin type (i.e. of type A_n, D_n, E_6, E_7, E_8), then all thick subcategories are admissible.

Corollary

Let Q be of Dynkin type and X, Y in $\mathrm{D}^b(\mathrm{mod}\, kQ)$. Then

$$\mathrm{Hom}^*(X, Y) = 0 \iff \mathrm{cox}(X) \leq \mathrm{cox}(Y)^{-1} c.$$

Example: coherent sheaves on $\mathbb{P}^1_k$

Fix a field k and let $\mathbb{P}^1_k$ denote the projective line over k. We consider the derived category $\mathsf{T} = \mathsf{D}^b(\operatorname{coh} \mathbb{P}^1_k)$.

Proposition (Beĭlinson, 1978)

There is a triangle equivalence

$$\mathsf{D}^b(\operatorname{coh} \mathbb{P}^1_k) \xrightarrow{\sim} \mathsf{D}^b(\operatorname{mod} kQ)$$

where Q denotes the Kronecker quiver $\circ \rightrightarrows \circ$.

- The thick tensor ideals of T are parameterised by subsets of the set of closed points $\mathbb{P}^1(k)$ [Thomason, 1997].
- The admissible thick subcategories of T are parameterised by non-crossing partitions.
- A non-trivial thick subcategory of T is either tensor ideal or admissible, but not both.

Concluding remarks

- We have seen some classification results for thick and localising subcategories of triangulated categories.
- There is a well developed theory for tensor triangulated categories or catgeories with an R-linear action.
- Is there unifying approach (support theory) to capture classifications via cohomology (tensor ideals) and exceptional sequences (admissible subcategories)?
- Do localising subcategories form a set? This is not even known for $\mathsf{D}(\mathsf{Qcoh}\,\mathbb{P}^1_k)$.
- A compactly generated triangulated category T admits a canonical filtration

$$\mathsf{T} = \bigcup_{\kappa \text{ regular}} \mathsf{T}^{\kappa}.$$

Can we classify κ-localising subcategories for $\kappa > \omega$?

With my coauthors at Oberwolfach in 2010