COMPLETIONS OF TRIANGULATED CATEGORIES

HENNING KRAUSE

Abstract. Notes for a master class at Aarhus University (March 22–24, 2023).

Contents

0. Introduction

Completions arise in all parts of mathematics. For example, the real numbers are the completion of the rationals. This construction uses equivalence classes of Cauchy sequences and is due to Cantor [4] and Méray [17]. There is an obvious generalisation in the setting of metric spaces, leading to the completion of a metric space. Completions of rings and modules play an important role in algebra and geometry. For categories there is the notion of ind-completion due to Grothendieck and Verdier [8] which provides an embedding into categories with filtered colimits.

In these notes we combine all these ideas to study completions of triangulated categories. We do not offer an elaborated theory and rather look at various examples which arise naturally in representation theory of algebras.

These notes are based on three lectures; they are divided into nine sections (roughly three per lecture). The first three sections provide some preparations. In particular, we introduce the ind-completion of a category and explain completions in the context of modules over commutative rings. In §4 we propose a definition of a partial completion for triangulated categories; roughly speaking these are full subcategories of the ind-completion which admit a triangulated structure. In §5 we provide various criteria for an exact functor between triangulated categories to be a partial completion, and §6 discusses a general set-up in terms of compactly generated triangulated categories which covers many of the examples that arise in nature. The final sections provide detailed proofs for some of the examples and discuss the role of enhancements.

It is my pleasure to thank Charley Cummings, Sira Gratz, and Davide Morigi for organising the *Categories, clusters, and completions master class* at Aarhus

Date: March 29, 2023.

University. In particular, I am very grateful to them for suggesting the topic of this series of lectures. My own interest in this subject arose from my collaboration with Dave Benson, Srikanth Iyengar, and Julia Pevtsova; I am most grateful for their inspiration and for their helpful comments on these notes. Last but not least let me thank Amnon Neeman for his comments, and let me recommend the notes from his lectures for another perspective on this fascinating subject.

1. The ind-completion of a category

Let $\mathcal{C}$ be an essentially small category. We write $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ for the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Morphisms in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ are natural transformations. Note that this category is complete and cocomplete, that is, all limits and colimits exist and are computed pointwise in Set.

For functors E and F we write $\operatorname{Hom}(E, F)$ for the set of morphisms from E to F. The *Yoneda functor*

$$\mathcal{C} \longrightarrow \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \quad X \mapsto H_X := \operatorname{Hom}(-, X)$$

is fully faithful; this follows from Yoneda's lemma which provides a natural bijection

$$\operatorname{Hom}(H_X, F) \xrightarrow{\sim} F(X).$$

Any functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ can be written canonically as a colimit of representable functors

$$\operatorname*{colim}_{H_X \to F} H_X \xrightarrow{\sim} F \tag{1.1}$$

where the colimit is taken over the *slice category* $\mathcal{C}/F$; see [8, Proposition 3.4]. Objects in $\mathcal{C}/F$ are morphisms $H_X \to F$ where X runs through the objects of $\mathcal{T}$. A morphism in $\mathcal{C}/F$ from $H_X \xrightarrow{\phi} F$ to $H_{X'} \xrightarrow{\phi'} F$ is a morphism $\alpha\colon X \to X'$ in $\mathcal{C}$ such that $\phi' H_\alpha = \phi$.

Definition 1.2. A *filtered colimit* in a category $\mathcal{D}$ is the colimit of a functor $\mathcal{I} \to \mathcal{D}$ such that the category $\mathcal{I}$ is *filtered*, that is

(1) the category is non-empty,
(2) given objects i, i' there is an object j with morphisms $i \to j \leftarrow i'$, and
(3) given morphisms $\alpha, \alpha'\colon i \to j$ there is a morphism $\beta\colon j \to k$ such that $\beta\alpha = \beta\alpha'$.

Remark 1.3. A partially ordered set $(I, \le)$ can be viewed as a category: the objects are the elements of I and there is a unique morphism $i \to j$ whenever $i \le j$. This category is filtered if and only if $(I, \le)$ is non-empty and *directed*, that is, for each pair of elements i, i' there is an element j such that $i, i' \le j$. When the colimit of a functor $\mathcal{I} \to \mathcal{D}$ is given by a directed partially ordered set, this colimit is also called *directed colimit* (or confusingly *direct limit*).

For each essentially small filtered category $\mathcal{I}$ there exists a functor $\phi\colon \mathcal{J} \to \mathcal{I}$ such that $\mathcal{J}$ is the category corresponding to a directed partially ordered set and any functor $X\colon \mathcal{I} \to \mathcal{D}$ induces an isomorphism

$$\operatorname*{colim}_{j \in \mathcal{J}} X(\phi(j)) \xrightarrow{\sim} \operatorname*{colim}_{i \in \mathcal{I}} X(i).$$

This fact will not be needed, but it may be useful to know; see [8, Proposition 8.1.6].

Let us get back to functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. It is not difficult to show that F is a filtered colimit of representable functors if and only if the slice category $\mathcal{C}/F$ is filtered.

Definition 1.4. The *ind-completion* of $\mathcal{C}$ is the category of functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ that are filtered colimits of representable functors. We denote this category by $\operatorname{Ind}\mathcal{C}$; it is a category with filtered colimits and the Yoneda functor $\mathcal{C} \to \operatorname{Ind}\mathcal{C}$ is the universal functor from $\mathcal{C}$ to a category with filtered colimits.

It is convenient to identify $\mathcal{C}$ with the full subcategory of representable functors in $\operatorname{Ind}\mathcal{C}$. Let $X = \operatorname{colim}_i X_i$ and $Y = \operatorname{colim}_j Y_j$ be objects in $\operatorname{Ind}\mathcal{C}$, written as filtered colimits of objects in $\mathcal{C}$.

Lemma 1.5. *We have natural bijections*

$$\operatorname*{colim}_i \operatorname{Hom}(C, X_i) \xrightarrow{\sim} \operatorname{Hom}(C, \operatorname*{colim}_i X_i) \quad \textit{for each} \quad C \in \mathcal{C}$$

and

$$\operatorname{Hom}(X,Y) \xrightarrow{\sim} \lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j).$$

Proof. The first bijection is an immediate consequence of Yoneda's lemma. For the second bijection we compute

$$\begin{aligned}\operatorname{Hom}(X,Y) &= \operatorname{Hom}(\operatorname*{colim}_i X_i, \operatorname*{colim}_j \operatorname{Hom} Y_j)\\ &\cong \lim_i \operatorname{Hom}(X_i, \operatorname*{colim}_j Y_j)\\ &\cong \lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j).\end{aligned}$$

□

The ind-completion takes a more familar form when we consider additive categories.

Example 1.6. Let A be a ring and $\mathcal{C} = \operatorname{proj} A$ the category of finitely generated projective A-modules. A theorem of Lazard says that a module is flat if and only if it is a filtered colimit of finitely generated free modules. Thus $\operatorname{Ind}\mathcal{C}$ identifies with the category of flat A-modules.

Example 1.7. Let A be a ring and $\mathcal{C} = \operatorname{mod} A$ the category of finitely presented A-modules. It is well known that any module is is a filtered colimit of finitely presented modules. Thus $\operatorname{Ind}\mathcal{C}$ identifies with the category of all A-modules.

Here is a useful fact which connects the above examples; it is due to Lenzing [15, Proposition 2.1]. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. The inclusion induces a fully faithful functor $\operatorname{Ind}\mathcal{D} \to \operatorname{Ind}\mathcal{C}$; this follows easily from Lemma 1.5. Thus we may view $\operatorname{Ind}\mathcal{D}$ as a full subcategory of $\operatorname{Ind}\mathcal{C}$.

Lemma 1.8. *An object $X \in \operatorname{Ind}\mathcal{C}$ belongs to $\operatorname{Ind}\mathcal{D}$ if and only if each morphism $C \to X$ with $C \in \mathcal{C}$ factors through an object in $\mathcal{D}$.*

Proof. If X belongs to $\operatorname{Ind}\mathcal{D}$, then it follows from Lemma 1.5 that each morphism from $C \in \mathcal{C}$ factors through an object in $\mathcal{D}$. Now write $X = \operatorname{colim}_{C \to X} C$ as a filtered colimit of objects in $\mathcal{C}$ as in (1.1), using the slice category $\mathcal{C}/X$ as index category. The condition that each morphism $C \to X$ with $C \in \mathcal{C}$ factors through an object in $\mathcal{D}$ is precisely the fact that $\mathcal{D}/X$ is a *cofinal subcategory* of $\mathcal{C}/X$. Thus the induced morphism

$$\operatorname*{colim}_{D \to X} D \longrightarrow \operatorname*{colim}_{C \to X} C = X$$

is an isomorphism; see [8, Proposition 8.1.3]. □

The above criterion applied to the inclusion $\operatorname{proj} A \subseteq \operatorname{mod} A$ shows that a module is flat if and only if each morphism from a finitely presented module factors through a finitely generated projective module.

Example 1.9. Let $\mathcal{C}$ be an essentially small additive category with cokernels. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is *left exact* if it is additive and sends each cokernel sequence $X \to Y \to Z \to 0$ to an exact sequence $0 \to FZ \to FY \to FX$ of abelian groups. One can show that F is left exact if and only if it is a filtered colimit of representable functors; see [14, Lemma 11.1.14]. Thus $\operatorname{Ind}\mathcal{C}$ identifies with the category $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. If $\mathcal{C}$ is abelian, then $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an abelian Grothendieck category and the Yoneda functor is exact.

Example 1.10. Let A be a commutative noetherian ring. Let $\mathcal{C} = \operatorname{fl} A$ denote the category of finite length modules (i.e. the finitely generated torsion modules). Then $\operatorname{Ind}\mathcal{C}$ identifies with the category of all torsion A-modules. This follows for example by applying the criterion in Lemma 1.8 to the inclusion $\operatorname{fl} A \subseteq \operatorname{mod} A$.

The next example is relevant because we wish to study filtered colimits arising from (or even in) triangulated categories.

Example 1.11. Let $\mathcal{C}$ be an essentially small triangulated category. Then $\operatorname{Ind}\mathcal{C}$ identifies with the category $\operatorname{Coh}\mathcal{C}$ of cohomological functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. Recall that $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is *cohomological* if it is additive and sends exact triangles to exact sequences.

Proof. Any representable functor is cohomological, and taking filtered colimits (in the category Ab) is exact. Thus a filtered colimit of representable functors is cohomological. Conversely, suppose that $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological. Then it easily checked that the slice category $\mathcal{C}/F$ is filtered. $\square$

2. The sequential completion of a category

The ind-completion of a category allows one to take arbitrary filtered colimits, so colimits of functors that are indexed by any filtered category. In the following we restrict to colimits of functors (or sequences) that are indexed by the natural numbers.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers, viewed as a category with a single morphism $i \to j$ if $i \leq j$.

Now fix a category $\mathcal{C}$ and consider the category $\operatorname{Fun}(\mathbb{N}, \mathcal{C})$ of functors $\mathbb{N} \to \mathcal{C}$. An object X is nothing but a sequence of morphisms $X_0 \to X_1 \to X_2 \to \cdots$ in $\mathcal{C}$, and the morphisms between functors are by definition the natural transformations. We call X a *Cauchy sequence* if for all $C \in \mathcal{C}$ the induced map $\operatorname{Hom}(C, X_i) \to \operatorname{Hom}(C, X_{i+1})$ is invertible for $i \gg 0$. This means:

$$\forall\, C \in \mathcal{C}\ \ \exists\, n_C \in \mathbb{N}\ \ \forall\, j \geq i \geq n_C \quad \operatorname{Hom}(C, X_i) \xrightarrow{\sim} \operatorname{Hom}(C, X_j).$$

Let $\operatorname{Cau}(\mathbb{N}, \mathcal{C})$ denote the full subcategory consisting of all Cauchy sequences. A morphism $X \to Y$ is *eventually invertible* if for all $C \in \mathcal{C}$ the induced map $\operatorname{Hom}(C, X_i) \to \operatorname{Hom}(C, Y_i)$ is invertible for $i \gg 0$. This means:

$$\forall\, C \in \mathcal{C}\ \ \exists\, n_C \in \mathbb{N}\ \ \forall\, i \geq n_C \quad \operatorname{Hom}(C, X_i) \xrightarrow{\sim} \operatorname{Hom}(C, Y_i).$$

Let S denote the class of eventually invertible morphisms in $\operatorname{Cau}(\mathbb{N}, \mathcal{C})$.

Definition 2.1. The *sequential Cauchy completion* of $\mathcal{C}$ is the category

$$\operatorname{Ind}_{\mathrm{Cau}}\mathcal{C} := \operatorname{Cau}(\mathbb{N}, \mathcal{C})[S^{-1}]$$

that is obtained from the Cauchy sequences by formally inverting all eventually invertible morphisms, together with the *canonical functor* $\mathcal{C} \to \operatorname{Ind}_{\mathrm{Cau}}\mathcal{C}$ that sends an object X in $\mathcal{C}$ to the constant sequence $X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} \cdots$.

A sequence $X\colon \mathbb{N}\to\mathcal{C}$ induces a functor

$$\widetilde{X}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad C\mapsto \operatorname*{colim}_i \operatorname{Hom}(C,X_i),$$

and this yields a functor

$$\operatorname{Ind}_{\mathrm{Cau}}\mathcal{C} \longrightarrow \operatorname{Ind}\mathcal{C} \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set}), \quad X\mapsto \widetilde{X},$$

because the assignment $X\mapsto\widetilde{X}$ maps eventually invertible morphisms to isomorphisms.

Proposition 2.2 ([13, Proposition 2.4])**.** *The canonical functor* $\operatorname{Ind}_{\mathrm{Cau}}\mathcal{C}\to\operatorname{Ind}\mathcal{C}$ *is fully faithful; it identifies* $\operatorname{Ind}_{\mathrm{Cau}}\mathcal{C}$ *with the colimits of sequences of representable functors that correspond to Cauchy sequences in* $\mathcal{C}$*.* □

It turns out that the class of Cauchy sequences is too restrictive; we need a more general notion of completion.

Definition 2.3. The *sequential completion* of $\mathcal{C}$ has as objects all sequences in $\mathcal{C}$, i.e. functors $\mathbb{N}\to\mathcal{C}$, and for objects X,Y set

$$\operatorname{Hom}(X,Y)=\lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i,Y_j).$$

This category is denoted by $\operatorname{Ind}_{\mathbb{N}}\mathcal{C}$, and for any class $\mathcal{X}$ of objects the corresponding full subcategory is denoted by $\operatorname{Ind}_{\mathcal{X}}\mathcal{C}$.

It follows from Lemma 1.5 that the assignment $X\mapsto \operatorname{colim}_i\operatorname{Hom}(-,X_i)$ induces a fully faithful functor $\operatorname{Ind}_{\mathbb{N}}\mathcal{C}\to\operatorname{Ind}\mathcal{C}$. Thus we obtain canonical inclusions

$$\operatorname{Ind}_{\mathcal{X}}\mathcal{C}\subseteq\operatorname{Ind}_{\mathbb{N}}\mathcal{C}\subseteq\operatorname{Ind}\mathcal{C}.$$

Example 2.4. Let $\mathcal{C}$ be an exact category and let $\mathcal{X}$ denote the class of sequences X such that each $X_i\to X_{i+1}$ is an admissible monomorphism. Then $\mathcal{C}^{\sim}:=\operatorname{Ind}_{\mathcal{X}}\mathcal{C}$ admits a canonical exact structure and is called *countable envelope* of $\mathcal{C}$ [9, Appendix B].

3. Completion of rings and modules

Let A be an associative ring. We consider the category $\operatorname{Mod}A$ of right A-modules and the following full subcategories:

$$\begin{aligned}
\operatorname{mod}A &= \text{finitely presented } A\text{-modules}\\
\operatorname{proj}A &= \text{finitely generated projective } A\text{-modules}\\
\operatorname{noeth}A &= \text{noetherian } A\text{-modules (satisfying the ascending chain condition)}\\
\operatorname{art}A &= \text{artinian } A\text{-modules (satisfying the descending chain condition)}\\
\operatorname{fl}A &= \operatorname{art}A\cap\operatorname{noeth}A = \text{finite length } A\text{-modules}
\end{aligned}$$

Definition 3.1. For an ideal $I\subseteq A$ the *I-adic completion* of A is the limit

$$\widehat{A}:=\lim_{n\geq 0} A/I^n.$$

Similarly for an A-module M one sets

$$\widehat{M}:=\lim_{n\geq 0} M/MI^n.$$

Let us consider the special case that A is a commutative noetherian local ring and $I=\mathfrak{m}$ its unique maximal ideal. We denote by $E=E(A/\mathfrak{m})$ the injective envelope of the unique simple A-module. Then $D=\operatorname{Hom}(-,E)$ yields the *Matlis duality* $\operatorname{Mod}A\to\operatorname{Mod}A$, satisfying

$$\operatorname{Hom}(M,DN)\cong\operatorname{Hom}(N,DM) \qquad (M,N\in\operatorname{Mod}A).$$

Note that $M \xrightarrow{\sim} D^2M$ when M has finite length; this is easily checked by induction on the length of M. Thus D induces an equivalence

$$(\operatorname{fl} A)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{fl} A.$$

For $n \geq 0$ we write $E_n = \operatorname{Hom}(A/\mathfrak{m}^n, E)$ and note that

$$E = \bigcup_{n\geq 0} E_n.$$

In fact, the module E is artinian and each submodule E_n is of finite length. Thus

$$\operatorname{Hom}(E,E) \cong \operatorname{Hom}(\operatorname*{colim}_{n\geq 0} E_n, E) \cong \lim_{n\geq 0} \operatorname{Hom}(E_n, E) \cong \lim_{n\geq 0} A/\mathfrak{m}^n = \widehat{A}.$$

In particular, each Matlis dual module DM is canonically an $\widehat{A}$-module via the map $\widehat{A} \xrightarrow{\sim} \operatorname{End}(E)$. Thus Matlis duality yields the following commutative diagram.

$$\begin{tikzcd}
(\operatorname{fl} A)^{\mathrm{op}} \arrow[rr, tail] \arrow[d, "\wr"] & & (\operatorname{art} A)^{\mathrm{op}} \arrow[d, "D"', "\wr"] \\
\operatorname{fl} A \arrow[r, tail] & \operatorname{noeth} A \arrow[r, "\widehat{\ }"] & \operatorname{noeth} \widehat{A}
\end{tikzcd}$$

Given a module M, the *socle* $\operatorname{soc} M$ is the sum of all simple submodules. One defines inductively $\operatorname{soc}^n M \subseteq M$ for $n \geq 0$ by setting $\operatorname{soc}^0 M = 0$, and $\operatorname{soc}^{n+1} M$ is given by the exact sequence

$$0 \longrightarrow \operatorname{soc}^n M \longrightarrow \operatorname{soc}^{n+1} M \longrightarrow \operatorname{soc}(M/\operatorname{soc}^n M) \longrightarrow 0.$$

Recall that a ring A is *semi-local* if $A/J(A)$ is a semisimple ring, where $J(A)$ denotes the Jacobson radical of A. In that case we have $\operatorname{soc} M \cong \operatorname{Hom}(A/J(A), M)$ for every A-module M.

Proposition 3.2 ([13, Proposition 3.4])**.** *Let A be a commutative noetherian semi-local ring. Then the sequential Cauchy completion of $\operatorname{fl} A$ identifies with $\operatorname{art} A$.*

Proof. Set $\mathcal{C} = \operatorname{fl} A$. The assignment $X \mapsto \bar{X} := \operatorname{colim}_n X_n$ yields a fully faithful functor

$$\operatorname{Ind}_{\mathrm{Cau}} \mathcal{C} \subseteq \operatorname{Ind} \mathcal{C} \longrightarrow \operatorname{Ind}(\operatorname{mod} A) = \operatorname{Mod} A.$$

It is well known that an A-module M is artinian if and only if M is the union of finite length modules and $\operatorname{soc} M$ has finite length [14, Proposition 2.4.20]. In that case the socle series $(\operatorname{soc}^n M)_{n\geq 0}$ of M yields a Cauchy sequence in $\mathcal{C}$ with $\operatorname{colim}_n(\operatorname{soc}^n M) = M$.

Now let $X \in \operatorname{Ind}_{\mathrm{Cau}} \mathcal{C}$. Then every finitely generated submodule of $\bar{X}$ has finite length, so $\bar{X}$ is a union of finite length modules. Also $\operatorname{soc} \bar{X}$ has finite length, since

$$\operatorname{soc} \bar{X} \cong \operatorname{Hom}(A/J(A), \bar{X}) \cong \operatorname*{colim}_n \operatorname{Hom}(A/J(A), X_n).$$

Thus $\bar{X}$ is artinian. □

Remark 3.3. Completions of modules or categories of modules will serve as model for completions of triangulated categories. We have seen two types of completions: the *ind-completion* (the inclusion $\operatorname{fl} A \to \operatorname{art} A$) and the *adic completion* (the functor $\operatorname{noeth} A \to \operatorname{noeth} \widehat{A}$). Both types of completions have their analogues when we consider triangulated categories.

4. Completion of triangulated categories

We are ready to propose a definition of 'completion' for a triangulated category. Roughly speaking it is a triangulated approximation of the ind-completion. In fact, it will be rare that the ind-completion of a triangulated category admits a triangulated structure, but it does happen that certain full subcategories are triangulated.

Let $\mathcal{C}$ be an essentially small triangulated category. We denote by $\operatorname{Coh}\mathcal{C}$ the category of cohomological functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. In Example 1.11 we have already seen that $\operatorname{Coh}\mathcal{C}$ equals $\operatorname{Ind}\mathcal{C}$. An exact functor $f\colon \mathcal{C} \to \mathcal{D}$ between triangulated categories induces the *restriction*

$$f_*\colon \mathcal{D} \longrightarrow \operatorname{Coh}\mathcal{C}, \quad X \mapsto \operatorname{Hom}(-, X) \circ f.$$

Definition 4.1. We call a fully faithful exact functor $f\colon \mathcal{C} \to \mathcal{D}$ a *partial completion* of the triangulated category $\mathcal{C}$ if the restriction $f_*\colon \mathcal{D} \to \operatorname{Coh}\mathcal{C}$ is fully faithful. The completion is called *sequential* if the above functor factors through $\operatorname{Ind}_{\mathbb{N}}\mathcal{C}$, and it is *Cauchy sequential* if the functor factors through $\operatorname{Ind}_{\mathrm{Cau}}\mathcal{C}$.

A partial completion of $\mathcal{C}$ is far from unique. But depending on the context there are often natural choices. An essential feature of a partial completion is the fact that any object in $\mathcal{D}$ can be written canonically as a filtered colimit of objects in the image of f. Suppose for simplicity that f is an inclusion. Then we have for each object $X \in \mathcal{D}$ an isomorphism

$$\operatorname*{colim}_{C \to X} C \xrightarrow{\sim} X$$

where $C \to X$ runs through all morphisms in $\mathcal{D}$ with $C \in \mathcal{C}$. This follows from (1.1). Moreover, using Lemma 1.5 we can compute morphisms in $\mathcal{D}$ via

$$\operatorname{Hom}(X, X') \cong \lim_{C \to X} \operatorname*{colim}_{C' \to X'} \operatorname{Hom}(C, C').$$

If the completion is sequential, then there is for each object $X \in \mathcal{D}$ a sequence $C_0 \to C_1 \to C_2 \to \cdots$ in $\mathcal{C}$ such that

$$\operatorname*{colim}_{n} C_n \xrightarrow{\sim} X.$$

We note that these filtered colimits are taken in $\mathcal{D}$; they exist because $\mathcal{D}$ identifies with a full subcategory of the ind-completion of $\mathcal{C}$.

Examples of partial completions arise from derived categories of exact subcategories. For an exact category $\mathcal{A}$ we write $\mathbf{D}(\mathcal{A})$ for its derived category and $\mathbf{D}^b(\mathcal{A})$ for the full subcategory of bounded complexes.

Example 4.2. For a commutative noetherian ring A the inclusion $\mathbf{D}^b(\operatorname{fl} A) \to \mathbf{D}^b(\operatorname{art} A)$ is a sequential partial completion.

Recall that a ring A is *right coherent* if the category $\operatorname{mod} A$ of finitely presented A-modules is abelian.

Example 4.3. For a right coherent ring A the inclusion $\mathbf{D}^b(\operatorname{proj} A) \to \mathbf{D}^b(\operatorname{mod} A)$ is a Cauchy sequential partial completion.

The assumption on the ring A to be right coherent is not essential. For an arbitrary ring one takes instead of $\operatorname{mod} A$ the exact category of A-modules M that admit a projective resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is finitely generated.

We will return to these examples and provide full proofs. In fact, the proofs require the study of compactly generated triangulated categories.

5. Completion of compact objects and pure-injectivity

Partial completions of an essentially small triangulated category $\mathcal{C}$ often arise as full triangulated subcategories of a compactly generated triangulated category $\mathcal{T}$ such that $\mathcal{C}$ equals the subcategory of compact objects.

Let $\mathcal{T}$ be a triangulated category that admits arbitrary coproducts. An object X in $\mathcal{T}$ is called *compact* if the functor $\operatorname{Hom}(X,-)$ preserves all coproducts. We denote by $\mathcal{T}^c$ the full subcategory of compact objects and note that it is a thick subcategory of $\mathcal{T}$. The triangulated category $\mathcal{T}$ is *compactly generated* if $\mathcal{T}^c$ is essentially small and if $\mathcal{T}$ has no proper localising subcategory containing $\mathcal{T}^c$.

From now on fix a compactly generated triangulated category $\mathcal{T}$ and set $\mathcal{C} = \mathcal{T}^c$.

Definition 5.1. The functor

$$\mathcal{T} \longrightarrow \operatorname{Coh} \mathcal{C}, \quad X \mapsto h_X := \operatorname{Hom}(-,X)|_{\mathcal{C}}$$

is called *restricted Yoneda functor*. The induced map

$$\operatorname{Hom}(X,Y) \longrightarrow \operatorname{Hom}(h_X,h_Y) \qquad (X,Y \in \mathcal{T})$$

is in general neither injective nor surjective; its kernel is the subgroup of *phantom morphisms*.

The category $\operatorname{Coh}\mathcal{C}$ is an extension closed subcategory of the abelian category $\operatorname{Mod}\mathcal{C}$ (i.e. the category of additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$). Thus it is an exact category (in the sense of Quillen) with enough projective and enough injective objects. In fact, the projective objects are of the form h_X with X a direct summand of a coproduct of compact objects in $\mathcal{T}$; this follows from Yoneda's lemma. An application of Brown's representability theorem shows that also the injective objects are of the form h_X for an object X in $\mathcal{T}$. This leads to the notion of a pure-injective object.

Definition 5.2. An exact triangle $X \to Y \to Z \to$ in $\mathcal{T}$ is called *pure-exact* if the induced sequence $0 \to h_X \to h_Y \to h_Z \to 0$ is exact. The triangle *splits* if $X \to Y$ is a split monomorphism, equivalently if $Y \to Z$ is a split epimorphism. An object X in $\mathcal{T}$ is *pure-injective* if each pure-exact triangle $X \to Y \to Z \to$ in $\mathcal{T}$ splits.

We have the following characterisation of pure-injectivity.

Proposition 5.3 ([10, Theorem 1.8])**.** *For an object X in $\mathcal{T}$ the following are equivalent.*

1. *The map* $\operatorname{Hom}(X',X) \to \operatorname{Hom}(h_{X'},h_X)$ *is bijective for all* $X' \in \mathcal{T}$.
2. *The object* h_X *is injective in* $\operatorname{Coh}\mathcal{C}$.
3. *The object* X *is pure-injective in* $\mathcal{T}$.
4. *For each set* I *the summation morphism* $\coprod_I X \to X$ *factors through the canonical morphism* $\coprod_I X \to \prod_I X$. □

The following immediate consequence motivates our interest in pure-injectives.

Corollary 5.4. *Let $\mathcal{D} \subseteq \mathcal{T}$ be a triangulated subcategory containing all compact objects and consisting of pure-injective objects. Then the inclusion $\mathcal{C} \to \mathcal{D}$ is a partial completion.* □

Pure-injectivity is a useful homological condition but in practice hard to check. In particular, there is no obvious triangulated structure on pure-injectives. We provide a criterion for a strong form of pure-injectivity which is an analogue of artinianess for modules.

Let $X \in \mathcal{T}$ and $C \in \mathcal{T}^c$. A *subgroup of finite definition* is a subgroup of $\operatorname{Hom}(C,X)$ that equals the image of an induced map $\operatorname{Hom}(D,X) \to \operatorname{Hom}(C,X)$ given by a morphism $C \to D$ in $\mathcal{T}^c$. Note that any subgroup of finite definition of $\operatorname{Hom}(C,X)$ is an $\operatorname{End}(X)$-submodule.

Lemma 5.5. *The subgroups of finite definition of* $\operatorname{Hom}(C,X)$ *are closed under finite sums and intersections. Thus they form a lattice.*

Proof. Let U_i be the image of $\operatorname{Hom}(D_i,X)\to\operatorname{Hom}(C,X)$ given by a morphism $C\to D_i$ in $\mathcal{T}^c$ $(i=1,2)$. Then U_1+U_2 equals the image of the map induced by $C\to D_1\oplus D_2$. Now complete this to an exact triangle $C\to D_1\oplus D_2\to E\to$. Then $U_1\cap U_2$ equals the image of the map induced by $C\to D_1\to E$. □

We say that an object $X\in\mathcal{T}$ satisfies *dcc on subgroups of finite definition* if for each compact object C any chain of subgroups of finite definition

$$\cdots\subseteq U_2\subseteq U_1\subseteq U_0=\operatorname{Hom}(C,X)$$

stabilises.

The following result goes back to Crawley-Boevey [5, 3.5], who proved this for locally finitely presented additive categories; see also [14, Theorem 12.3.4]. The proof is quite involved; the basic idea is to translate the descending chain condition into a noetherianess condition for some appropriate Grothendieck category (a localisation of $\operatorname{Mod}\mathcal{C}$ cogenerated by h_X).

Proposition 5.6. *For an object* $X\in\mathcal{T}$ *the following are equivalent.*

1. X *is* Σ*-pure-injective, i.e. any coproduct of copies of* X *is pure-injective.*
2. X *satisfies dcc on subgroups of finite definition.*
3. *Every product of copies of* X *decomposes into a coproduct of indecomposable objects with local endomorphism rings.*

Proof. The category $\operatorname{Coh}\mathcal{C}$ is locally finitely presented and has products; so Crawley-Boevey's theory can be applied. In particular, X is pure-injective in $\mathcal{T}$ if and only if h_X is pure-injective in $\operatorname{Coh}\mathcal{C}$, by Theorem 1 in [5, 3.5] and Proposition 5.3. Also, h_X satisfies dcc on subgroups of finite definition in $\operatorname{Coh}\mathcal{C}$ if and only if X satisfies dcc on subgroups of finite definition in $\mathcal{T}$, as $\operatorname{Hom}(C,X)\cong\operatorname{Hom}(h_C,h_X)$ for each compact object C. Now the assertion follows from Theorem 2 in [5, 3.5]. □

Let R be a commutative ring and suppose that $\mathcal{C}$ is R*-linear*. This means that there is a ring homomorphism $R\to Z(\mathcal{C})$ into the *centre* of $\mathcal{C}$ (the ring of natural transformations $\mathrm{id}_{\mathcal{C}}\to\mathrm{id}_{\mathcal{C}}$) so that for each pair of objects X,Y the group of morphisms $\operatorname{Hom}(X,Y)$ is naturally an R-module.

We view $\mathcal{C}$ as a full subcategory of $\operatorname{Coh}\mathcal{C}$ and call an object X in $\operatorname{Coh}\mathcal{C}$ or $\mathcal{T}$ *artinian* over R if $\operatorname{Hom}(C,X)$ is an artinian R-module for all $C\in\mathcal{C}$ (via the canonical homomorphism $R\to\operatorname{End}(C)$). Let $\operatorname{art}_R\mathcal{C}$ denote the full subcategory of R-artinian objects in $\operatorname{Coh}\mathcal{C}$. One may drop R when $R=Z(\mathcal{C})$.

Corollary 5.7. *The category* $\operatorname{art}_R\mathcal{C}$ *of* R*-artinian objects admits a canonical triangulated structure (induced from that of* $\mathcal{T}$*). Moreover, the inclusion* $\mathcal{C}\to\operatorname{art}_R\mathcal{C}$ *is a partial completion.*

Proof. The assertion follows from the fact that the R-artinian objects of $\operatorname{Coh}\mathcal{C}$ identify via the restricted Yoneda functor with the thick subcategory of R-artinian objects of $\mathcal{T}$, which consists of pure-injective objects. Clearly, if $X\in\mathcal{T}$ is R-artinian, then h_X is R-artinian. For the converse suppose that $X\in\operatorname{Coh}\mathcal{C}$ is R-artinian. We have for $X\in\operatorname{Coh}\mathcal{C}$ and $C\in\mathcal{C}$ the analogous concept of a subgroup of finite definition of $\operatorname{Hom}(C,X)$, and then artinianess over R implies dcc on subgroups of finite definition. Thus it follows from Theorem 2 in [5, 3.5] that X is pure-injective. The exact structure on $\operatorname{Coh}\mathcal{C}$ agrees with the pure-exact structure. Thus X is an injective object and therefore of the form $h_{\bar X}$ for a pure-injective object $\bar X$ in $\mathcal{T}$ by Proposition 5.3. Clearly, $\bar X$ is R-artinian. □

Corollary 5.8. *Let $\mathcal{D} \subseteq \mathcal{T}$ be a triangulated subcategory containing all compact objects and consisting of R-artinian objects. Then the inclusion $\mathcal{C} \to \mathcal{D}$ is a partial completion.*

Proof. The assertion is an immediate consequence of Corollary 5.4 since each object in $\mathcal{D}$ is Σ-pure-injective thanks to Proposition 5.6. □

Remark 5.9. There is a notion of a *locally finite* triangulated category; see [11, 21]. One way of defining this is that all cohomological functors into abelian groups (covariant or contravariant) are coproducts of direct summands of representable functors. An equivalent condition is that every short exact sequence of cohomological functors does split. Examples are the stable module category $\operatorname{stmod} A$ when A is a self-injective algebra of finite representation type, or the derived category $\mathbf{D}^b(\operatorname{mod} A)$ of a hereditary algebra of finite representation type. Then we have equivalences $\operatorname{Ind}(\operatorname{stmod} A) \simeq \operatorname{StMod} A$ and $\operatorname{Ind}(\mathbf{D}^b(\operatorname{mod} A)) \simeq \mathbf{D}(\operatorname{Mod} A)$. In particular, the ind-completions carry a triangulated structure where each pure-exact triangle splits.

Now suppose that $\mathcal{C}$ is an essentially small triangulated category that is not locally finite. For example, let $\mathcal{C} = \operatorname{stmod} A$ when A is a finite dimensional self-injective algebra of infinite representation type. Passing to $\mathcal{C}^{\mathrm{op}}$ if necessary this means that not all objects in $\operatorname{Ind}\mathcal{C}$ are projective. So we find an exact sequence $0 \to X \to Y \to Z \to 0$ which does not split. On the other hand, if $\operatorname{Ind}\mathcal{C}$ admits a triangulated structure, then each kernel-cokernel pair needs to split. It follows that $\operatorname{Ind}\mathcal{C}$ does not admit a triangulated structure.

6. Torsion versus completion

For any compactly generated triangulated category and any choice of compact objects generating a localising subcategory, there is an adjoint pair of functors that resembles derived torsion and completion functors for the derived category of a commutative ring.

Let $\mathcal{T}$ be a compactly generated triangulated category with suspension $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ and set $\mathcal{C} = \mathcal{T}^c$. We choose a thick subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ and denote by $\mathcal{T}_0 \subseteq \mathcal{T}$ the localising subcategory which is generated by $\mathcal{C}_0$. Note that $(\mathcal{T}_0)^c = \mathcal{C}_0$. The inclusion $\mathcal{T}_0 \to \mathcal{T}$ admits a right adjoint, by Brown's representability theorem, which we denote by $q\colon \mathcal{T} \to \mathcal{T}_0$. This functor preserves coproducts and then another application of Brown's representability theorem yields a right adjoint q_ρ. Thus the left adjoint q_λ and the right adjoint q_ρ provide two embeddings of $\mathcal{T}_0$ into $\mathcal{T}$, and our notation suggests a symmetry which does not give preference to any of the inclusions.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "q" description, two heads] & \mathcal{T}_0 \arrow[l, bend right, tail, "q_\lambda"'] \arrow[l, bend left, tail, "q_\rho"] \end{tikzcd}$$

Definition 6.1. The choice of $\mathcal{C}_0 \subseteq \mathcal{C}$ yields exact functors

$$\Gamma := q_\lambda \circ q \qquad \text{and} \qquad \Lambda := q_\rho \circ q$$

which form an adjoint pair. For any object X in $\mathcal{T}$ the unit $X \to \Lambda X$ is called *completion* and the counit $\Gamma X \to X$ is called *torsion* or *local cohomology* of X.

Note that these functors are idempotent, since $\mathrm{id}_{\mathcal{T}_0} \xrightarrow{\sim} q \circ q_\lambda$ and $q \circ q_\rho \xrightarrow{\sim} \mathrm{id}_{\mathcal{T}_0}$. From the definitions it is clear that the adjoint pair (Γ, Λ) induces mutually inverse equivalences

$$\begin{tikzcd} \Lambda\mathcal{T} \arrow[r, shift left, "\Gamma"] & \Gamma\mathcal{T} \arrow[l, shift left, "\Lambda"] \end{tikzcd} \tag{6.2}$$

where $\Lambda\mathcal{T} = \{X \in \mathcal{T} \mid X \xrightarrow{\sim} \Lambda X\}$ and $\Gamma\mathcal{T} = \{X \in \mathcal{T} \mid \Gamma X \xrightarrow{\sim} X\}$.

We are interested in the completions of the compact objects from $\mathcal{T}$, and we may view these as objects of $\Gamma\mathcal{T} = \mathcal{T}_0$, because of the above equivalence (6.2). Thus it is equivalent to look at the local cohomology of the compact objects from $\mathcal{T}$. Set

$$\widehat{\mathcal{C}} := \operatorname{thick}(\Gamma\mathcal{C}) \simeq \operatorname{thick}(\Lambda\mathcal{C}) \subseteq \Lambda\mathcal{T}.$$

Then we obtain the following chain of inclusions.

$$\mathcal{C}_0 = (\Gamma\mathcal{T})^c \subseteq \widehat{\mathcal{C}} \subseteq \Gamma\mathcal{T} \subseteq \mathcal{T}$$

The next diagram shows how these various subcategories of $\mathcal{T}$ are related. The inclusion $\mathcal{T}_0 \to \mathcal{T}$ admits the right adjoint Γ, and we may think of its restriction $\mathcal{C} \to \widehat{\mathcal{C}}$ as a mock right adjoint of the inclusion $\mathcal{C}_0 \to \mathcal{C}$.

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, tail] \arrow[d, tail] & \widehat{\mathcal{C}} \arrow[r, tail] & \mathcal{T}_0 \arrow[d, tail, shift right] \\
\mathcal{C} \arrow[ur] \arrow[rr, tail] & & \mathcal{T} \arrow[u, shift right, "\Gamma"']
\end{tikzcd}$$

Problem 6.3. Find a description of $\widehat{\mathcal{C}}$ for given triangulated categories $\mathcal{C}_0 \subseteq \mathcal{C}$.

Let us get back to the distinction between ind-completion and adic completion (cf. Remark 3.3). This carries over to our triangulated setting; it means that we can approach $\widehat{\mathcal{C}}$ from two directions, using either the inclusion $\mathcal{C}_0 \to \widehat{\mathcal{C}}$ or the functor $\mathcal{C} \to \widehat{\mathcal{C}}$.

We illustrate this with an example which explains the terminology; it goes back to work of Dwyer, Greenlees, and May [6, 7]. Let A be a commutative ring. We set $\mathcal{T} = \mathbf{D}(\operatorname{Mod} A)$ and identify $\mathbf{D}^b(\operatorname{proj} A) = \mathcal{T}^c = \mathcal{C}$ (the category of perfect complexes). Recall for an ideal $I \subseteq A$ and any A-module M the definition of *I-torsion*

$$M \longmapsto \operatorname*{colim}_{n\geq 0}(\operatorname{Hom}_A(A/I^n, M)) \subseteq M$$

and *I-adic completion*

$$M \longmapsto \lim_{n\geq 0}(M \otimes_A A/I^n).$$

Example 6.4 ([6])**.** Fix a finitely generated ideal $I \subseteq A$ and let $\mathcal{C}_0$ denote the category of perfect complexes having I-torsion cohomology. Then $\mathcal{T}_0$ equals the category of all complexes in $\mathcal{T}$ having I-torsion cohomology. The functor Γ equals the local cohomology functor (i.e. the right derived functor of I-torsion), while Λ equals the derived completion functor (i.e. the left derived functor of I-adic completion). Moreover, $\widehat{\mathcal{C}}$ is triangle equivalent to $\mathbf{D}^b(\operatorname{proj} \widehat{A})$.

Proof. Let K denote the Koszul complex given by a finite sequence of generators of I. We view K as a dg left module over the dg endomorphism ring $E = \mathcal{E}nd_A(K)$ and we view $K^\vee = \mathcal{H}om_A(K, A)$ as a dg right module over E. Let $\mathbf{D}(E)$ denote the derived category of the category of dg right E-modules. Then we obtain the following diagram

$$\begin{tikzcd}
\mathcal{T} = \mathbf{D}(A) \arrow[r, "q" description] & \mathbf{D}(E) \arrow[l, bend right, "q_\lambda"'] \arrow[l, bend left, "q_\rho"]
\end{tikzcd}$$

where

$$q = \mathcal{H}om_A(K, -) = - \otimes_A K^\vee \qquad q_\lambda = - \otimes_E K \qquad q_\rho = \mathcal{H}om_E(K^\vee, -).$$

In [6, §6] it is shown that q_λ identifies $\mathbf{D}(E)$ with the category of all complexes in $\mathbf{D}(A)$ having I-torsion cohomology, while q_ρ identifies $\mathbf{D}(E)$ with the category of all complexes in $\mathbf{D}(A)$ that are I-complete. Moreover, it is shown that $\Gamma = q_\lambda \circ q$

computes local cohomology, while $\Lambda = q_\rho \circ q$ yields derived completion. Next we compute the graded endomorphisms of $q(A) = K^\vee$ and have

$$\operatorname{Hom}^*_E(K^\vee, K^\vee) \cong \operatorname{Hom}^*_A(\Gamma A, \Gamma A) \cong \operatorname{Hom}^*_A(A, \Lambda A) \cong H^*(\Lambda A) \cong \widehat{A},$$

where the first isomorphism is induced by q_λ and the second comes from adjunction, using that $\Lambda\Gamma \cong \Lambda$. It follows that

$$\widehat{\mathcal{C}} = \operatorname{thick}(K^\vee) \simeq \mathbf{D}^b(\operatorname{proj} \widehat{A}). \qquad \square$$

We have a more specific description of $\widehat{\mathcal{C}}$ when A is local; it uses the tensor triangulated structure of the derived category of a commutative ring.

Example 6.5 ([1])**.** Let A be a commutative noetherian local ring and $\mathfrak{m}$ its maximal ideal. Let $\mathcal{C}_0$ denote the category of perfect complexes having $\mathfrak{m}$-torsion cohomology. Then $\widehat{\mathcal{C}}$ equals the subcategory of dualisable (or rigid) objects in the tensor triangulated category $\mathcal{T}_0$.

Let us provide a criterion for the functor $\mathcal{C}_0 \to \widehat{\mathcal{C}}$ to be a partial completion of triangulated categories; it covers the previous example of a local ring with $\mathcal{C}_0$ the category of perfect complexes having torsion cohomology. Our motivation is the following. If $\mathcal{C}_0 \to \widehat{\mathcal{C}}$ is a partial completion, then the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ taking X to ΓX induces a functor

$$\operatorname{Coh} \mathcal{C}_0 \supseteq \{\operatorname{Hom}(-, X)|_{\mathcal{C}_0} \mid X \in \mathcal{C}\} \xrightarrow{\gamma} \widehat{\mathcal{C}}$$

such that

1. γ is fully faithful and almost an equivalence, up to the fact that $\Gamma\mathcal{C} \subseteq \Gamma\mathcal{T}$ need not be a thick subcategory, and
2. the category $\{\operatorname{Hom}(-, X)|_{\mathcal{C}_0} \mid X \in \mathcal{C}\}$ is explicitly given by $\mathcal{C}_0 \subseteq \mathcal{C}$.

Proposition 6.6. *Let R be a commutative ring and suppose that $\mathcal{C}$ is R-linear. Suppose also that* $\operatorname{Hom}(X, Y)$ *is a finitely generated R-module for all objects X, Y in $\mathcal{C}$ and that* $\operatorname{End}(X)$ *has finite length for each X in $\mathcal{C}_0$. Then the inclusion $\mathcal{C}_0 \to \widehat{\mathcal{C}}$ is a partial completion.*

Proof. The assumption implies that for each $X \in \widehat{\mathcal{C}}$ and $C \in \mathcal{C}_0$ the R-module $\operatorname{Hom}(C, X)$ has finite length. Thus we can apply Corollary 5.8. $\square$

We continue with examples. Let A be a right coherent ring and denote by $\operatorname{Inj} A$ the category of all injective A-modules. Then the category of complexes $\mathbf{K}(\operatorname{Inj} A)$ (with morphisms the chain morphisms up to homotopy) is compactly generated, and taking a module to its injective resolution identifies $\mathbf{D}^b(\operatorname{mod} A)$ with the full subcategory of compact objects; see [14, Proposition 9.3.12] for the noetherian case and [12, Theorem 4.9] for the general case.

The following example should be compared with Example 4.3.

Example 6.7. Let A be a right coherent ring. We consider $\mathcal{T} = \mathbf{K}(\operatorname{Inj} A)$ and choose $\mathcal{C}_0 = \mathbf{D}^b(\operatorname{proj} A)$. Then it is easily checked that $\mathcal{C} \xrightarrow{\sim} \widehat{\mathcal{C}}$.

The next example is more challenging.

Example 6.8 (BG-conjecture)**.** Let k be a field and G a finite group. We consider the group algebra kG and set $\mathcal{T} = \mathbf{K}(\operatorname{Inj} kG)$. Let $\mathbf{i}k$ denote an injective resolution of the trivial representation. Then the assignment $X \mapsto X \otimes_k \mathbf{i}k$ identifies $\mathcal{C} = \mathbf{D}^b(\operatorname{mod} kG)$ with $\mathcal{T}^c$. We choose $\mathcal{C}_0 = \operatorname{thick}(k)$ and note that $\mathcal{T}_0$ identifies with the derived category of dg modules over the algebra $C^*(BG; k)$ (the cochains of the classifying space BG with coefficients in k) [2]. For $X \in \mathcal{T}$ we set

$$H^*(G, X) = \operatorname{Hom}^*(\mathbf{i}k, X) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}(\mathbf{i}k, \Sigma^n X)$$

and view this as a module over the cohomology algebra $H^*(G,k)$. It is well known that this algebra is noetherian and that $H^*(G,X)$ is a noetherian module for any compact object X. A recent (unpublished) conjecture of Benson and Greenlees asserts that $\widehat{\mathcal{C}}$ equals the category of objects in $\mathcal{T}_0$ such that $H^*(G,X)$ is a noetherian module over $H^*(G,k)$. Note that $\mathcal{C}_0 \to \widehat{\mathcal{C}}$ is a partial completion.

We offer another challenge.

Example 6.9 (Strong generation conjecture)**.** Let A be a commutative noetherian local ring. It is well known that A is regular if and only if the triangulated category $\mathbf{D}^b(\operatorname{proj} A)$ admits a strong generator (in the sense of Bondal and Van den Bergh [3]). On the other hand, A is regular if and only if its completion $\widehat{A}$ is regular. Keeping in mind Example 6.5, one may conjecture: If $\mathcal{C}$ admits a strong generator, then $\widehat{\mathcal{C}}$ admit a strong generator.

7. Completing complexes of finite length modules

We return to a previous example and give now a full proof of the following.

Proposition 7.1 ([13, Example 4.2])**.** *For a commutative noetherian ring A the inclusion*

$$\mathbf{D}^b(\operatorname{fl} A) \longrightarrow \mathbf{D}^b(\operatorname{art} A)$$

is a sequential partial completion.

Proof. Recall that an A-module M is artinian if and only if M is the union of finite length modules and $\operatorname{soc} M$ has finite length [14, Proposition 2.4.20]. In particular, the abelian category $\operatorname{art} A$ has enough injective objects.

We write $\operatorname{Mod}_0 A$ for the full subcategory of A-modules that are filtered colimits of finite length modules. Thus the inclusion $\operatorname{fl} A \to \operatorname{Mod} A$ induces an equivalence $\operatorname{Ind} \operatorname{fl} A \xrightarrow{\sim} \operatorname{Mod}_0 A$. Set

$$\operatorname{Inj}_0 A = \operatorname{Inj} A \cap \operatorname{Mod}_0 A \qquad \text{and} \qquad \operatorname{inj}_0 A = \operatorname{Inj} A \cap \operatorname{art} A.$$

Note that an injective A-module is in $\operatorname{Mod}_0 A$ if and only if each indecomposable direct summand is artinian. Now consider the compactly generated triangulated $\mathcal{T} = \mathbf{K}(\operatorname{Inj}_0 A)$ given by the complexes in $\operatorname{Inj}_0 A$. Then we have canonical triangle equivalences

$$\mathbf{D}^b(\operatorname{fl} A) \xrightarrow{\sim} \mathcal{T}^c \qquad \text{and} \qquad \mathbf{D}^b(\operatorname{art} A) \xrightarrow{\sim} \mathbf{K}^{+,b}(\operatorname{inj}_0 A) \subseteq \mathcal{T};$$

see Corollary 4.2.9 and Proposition 9.3.12 in [14]. Next observe that for $X \in \mathbf{D}^b(\operatorname{fl} A)$ and $Y \in \mathbf{D}^b(\operatorname{art} A)$ the A-module $\operatorname{Hom}(X,Y)$ has finite length. This amounts to showing that $\operatorname{Ext}^i(M,N)$ has finite length for all $M \in \operatorname{fl} A$, $N \in \operatorname{art} A$, and $i \in \mathbb{Z}$, which reduces to the case $i = 0$ by taking an injective resolution of N. Thus we can apply Corollary 5.8 and it follows that $\mathbf{D}^b(\operatorname{fl} A) \to \mathbf{D}^b(\operatorname{art} A)$ is a partial completion. It remains to observe that each complex X in $\mathbf{D}^b(\operatorname{art} A)$ is the colimit of the sequence $(\operatorname{soc}^n X)_{n\geq 0}$ in $\mathbf{D}^b(\operatorname{fl} A)$, but this need not be a Cauchy sequence. □

When the ring A is local and regular, the completion $\mathbf{D}^b(\operatorname{fl} A) \to \mathbf{D}^b(\operatorname{art} A)$ identifies with $\mathcal{C}_0 \to \widehat{\mathcal{C}}$ in Example 6.5.

8. Completing perfect complexes

We return to another example and give a full proof of the following.

Proposition 8.1 ([13, Theorem 6.2])**.** *For a right coherent ring A the inclusion*

$$\mathbf{D}^b(\operatorname{proj} A) \longrightarrow \mathbf{D}^b(\operatorname{mod} A)$$

is a Cauchy sequential partial completion.

The proof requires several lemmas which are of independent interest. Let $\mathcal{T}$ be a triangulated category with suspension $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ and suppose that countable coproducts exist in $\mathcal{T}$.

Definition 8.2. A *homotopy colimit* of a sequence of morphisms

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

in $\mathcal{T}$ is an object X that occurs in an exact triangle

$$\Sigma^{-1}X \longrightarrow \textstyle\coprod_{n\geq 0} X_n \xrightarrow{\mathrm{id}-\phi} \textstyle\coprod_{n\geq 0} X_n \longrightarrow X.$$

We write $\operatorname{hocolim}_n X_n$ for X and observe that a homotopy colimit is unique up to a (non-unique) isomorphism.

Recall that an object C in $\mathcal{T}$ is *compact* if $\operatorname{Hom}(C,-)$ preserves all coproducts. A morphism $X \to Y$ is *phantom* if any composition $C \to X \to Y$ with C compact is zero. The phantom morphisms form an ideal and we write $\operatorname{Ph}(X,Y)$ for the subgroup of all phantoms in $\operatorname{Hom}(X,Y)$.

Let us compute the functor $\operatorname{Hom}(-,\operatorname{hocolim}_n X_n)$. To this end observe that a sequence

$$A_0 \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$$

of maps between abelian groups induces an exact sequence

$$0 \longrightarrow \textstyle\coprod_{n\geq 0} A_n \xrightarrow{\mathrm{id}-\phi} \textstyle\coprod_{n\geq 0} A_n \longrightarrow \operatorname{colim}_n A_n \longrightarrow 0$$

because it identifies with the colimit of the exact sequences

$$0 \longrightarrow \textstyle\coprod_{i=0}^{n-1} A_i \xrightarrow{\mathrm{id}-\phi} \textstyle\coprod_{i=0}^{n} A_i \longrightarrow A_n \longrightarrow 0.$$

Lemma 8.3. *Let $C \in \mathcal{T}$ be compact. Then any sequence $X_0 \to X_1 \to X_2 \to \cdots$ in $\mathcal{T}$ induces an isomorphism*

$$\operatorname*{colim}_n \operatorname{Hom}(C,X_n) \xrightarrow{\sim} \operatorname{Hom}(C,\operatorname*{hocolim}_n X_n).$$

Proof. The above observation gives an exact sequence

$$0 \to \textstyle\coprod_n \operatorname{Hom}(C,X_n) \to \textstyle\coprod_n \operatorname{Hom}(C,X_n) \to \operatorname{colim}_n \operatorname{Hom}(C,X_n) \to 0.$$

Now apply $\operatorname{Hom}(C,-)$ to the defining triangle for $\operatorname{hocolim}_n X_n$. Comparing both sequences yields the assertion, since

$$\coprod_n \operatorname{Hom}(C,X_n) \cong \operatorname{Hom}(C,\coprod_n X_n). \qquad \square$$

Recall that for any sequence $\cdots \to A_2 \xrightarrow{\phi_2} A_1 \xrightarrow{\phi_1} A_0$ of maps between abelian groups the inverse limit and its first derived functor are given by the exact sequence

$$0 \to \lim_n A_n \to \textstyle\prod_{n\geq 0} A_n \xrightarrow{\mathrm{id}-\phi} \textstyle\prod_{n\geq 0} A_n \longrightarrow \lim^1_n A_n \longrightarrow 0.$$

Note that $\lim^1_n A_n = 0$ when $A_n \xrightarrow{\sim} A_{n+1}$ for $n \gg 0$.

The following result goes back to work of Milnor [18] and has been extended by several authors.

Lemma 8.4. *Let $X = \operatorname{hocolim}_n X_n$ be a homotopy colimit in $\mathcal{T}$ such that each X_n is a coproduct of compact objects. Then we have for any Y in $\mathcal{T}$ a natural exact sequence*

$$0 \to \operatorname{Ph}(X,Y) \to \operatorname{Hom}(X,Y) \to \lim_n \operatorname{Hom}(X_n,Y) \to 0$$

and an isomorphism

$$\operatorname{Ph}(X, \Sigma Y) \cong \lim_n{}^1 \operatorname{Hom}(X_n, Y).$$

Proof. Apply $\operatorname{Hom}(-, Y)$ to the exact triangle defining $\operatorname{hocolim}_n X_n$ and use that a morphism $X \to Y$ is phantom if and only if it factors through the canonical morphism $X \to \coprod_{n \geq 0} \Sigma X_n$. $\square$

Let $\mathcal{C} \subseteq \mathcal{T}$ be a full triangulated subcategory consisting of compact objects and consider the restricted Yoneda functor

$$\mathcal{T} \longrightarrow \operatorname{Coh} \mathcal{C}, \quad X \mapsto h_X := \operatorname{Hom}(-, X)|_{\mathcal{C}}.$$

This functor induces for each pair of objects $X, Y \in \mathcal{T}$ a map

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(h_X, h_Y).$$

Clearly, this map is bijective when X is in $\mathcal{C}$, and it remains bijective when X is a coproduct of objects in $\mathcal{C}$.

Lemma 8.5. *Let $X = \operatorname{hocolim}_n X_n$ be a homotopy colimit in $\mathcal{T}$ such that each X_n is a coproduct of objects in $\mathcal{C}$. Then we have for any Y in $\mathcal{T}$ a natural isomorphism*

$$\operatorname{Hom}(X, Y) / \operatorname{Ph}(X, Y) \xrightarrow{\sim} \operatorname{Hom}(h_X, h_Y).$$

Proof. We have

$$\begin{aligned} \operatorname{Hom}(X, Y) / \operatorname{Ph}(X, Y) &\cong \lim_n \operatorname{Hom}(X_n, Y) \\ &\cong \lim_n \operatorname{Hom}(h_{X_n}, h_Y) \\ &\cong \operatorname{Hom}(\operatorname{colim}_n h_{X_n}, h_Y) \\ &\cong \operatorname{Hom}(h_X, h_Y). \end{aligned}$$

The first isomorphism follows from Lemma 8.4, the second uses that each X_n is a coproduct of objects in $\mathcal{C}$, the third is clear, and the last follows from Lemma 8.3. $\square$

Proof of Proposition 8.1. Set $\mathcal{P} = \operatorname{proj} A$. The inclusion $\operatorname{proj} A \to \operatorname{mod} A$ induces a triangle equivalence

$$\mathbf{K}^{-,b}(\operatorname{proj} A) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} A).$$

Thus we may identify $\mathbf{D}^b(\operatorname{proj} A) \to \mathbf{D}^b(\operatorname{mod} A)$ with the inclusion

$$\mathbf{K}^b(\mathcal{P}) \longrightarrow \mathbf{K}^{-,b}(\mathcal{P}).$$

For any complex X we consider the sequence of truncations

$$\cdots \longrightarrow \sigma_{\geq n+1} X \longrightarrow \sigma_{\geq n} X \longrightarrow \sigma_{\geq n-1} X \longrightarrow \cdots$$

given by

$$\begin{tikzcd} \sigma_{\geq n} X \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X^n \arrow[r] \arrow[d, "\mathrm{id}"] & X^{n+1} \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\ X & \cdots \arrow[r] & X^{n-2} \arrow[r] & X^{n-1} \arrow[r] & X^n \arrow[r] & X^{n+1} \arrow[r] & \cdots . \end{tikzcd}$$

For X in $\mathbf{K}^{-,b}(\mathcal{P})$ and $n \in \mathbb{Z}$ we set $X_n = \sigma_{\geq -n} X$. This yields a Cauchy sequence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

in $\mathbf{K}^b(\mathcal{P})$ with $\operatorname{hocolim}_{n \geq 0} X_n \cong X$.

We claim that the restricted Yoneda functor

$$\mathbf{K}^{-,b}(\mathcal{P}) \longrightarrow \operatorname{Coh} \mathbf{K}^b(\mathcal{P}), \quad X \mapsto h_X := \operatorname{Hom}(-, X)|_{\mathbf{K}^b(\mathcal{P})},$$

is fully faithful. Let X, Y be objects in $\mathbf{K}^{-,b}(\mathcal{P})$. As before we write X as homotopy colimit of its truncations $X_n = \sigma_{\geq -n}X$ and denote by C_n the cone of $X_{n-1} \to X_n$. This complex is concentrated in degree $-n$; so $\operatorname{Hom}(C_n, Y) = 0$ for $n \gg 0$. Thus $X_n \to X_{n+1}$ induces a bijection

$$\operatorname{Hom}(X_{n+1}, Y) \xrightarrow{\sim} \operatorname{Hom}(X_n, Y) \quad \text{for} \quad n \gg 0.$$

This implies

$$\operatorname{Hom}(X, Y) \xrightarrow{\sim} \lim_n \operatorname{Hom}(X_n, Y)$$

and therefore $\operatorname{Ph}(X, Y) = 0$ by Lemma 8.4. From Lemma 8.5 we conclude that

$$\operatorname{Hom}(X, Y) \xrightarrow{\sim} \operatorname{Hom}(h_X, h_Y). \qquad \square$$

From the proof of Proposition 8.1 we learn that each complex in $\mathbf{D}^b(\operatorname{mod} A)$ is not only a filtered colimit of perfect complexes; it is actually the colimit of a Cauchy sequence which is obtained from its truncations. In particular, we are in the situation that a homotopy colimit is an honest colimit, and therefore unique up to a unique isomorphism.

9. Completion using enhancements

While the ind-completion of a category is a fairly explicit construction, it is not immediately clear how to deal with additional structure. In particular, there is no obvious triangulated structure for $\operatorname{Ind} \mathcal{C}$ when $\mathcal{C}$ is triangulated. One way to address this problem is the use of enhancements.

Recall that a triangulated category is *algebraic* if it is triangle equivalent to the stable category $\operatorname{St} \mathcal{A}$ of a Frobenius category $\mathcal{A}$. A morphism between exact triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

in $\operatorname{St} \mathcal{A}$ will be called *coherent* if it can be lifted to a morphism

$$\begin{tikzcd} 0 \arrow[r] & \tilde{X} \arrow[r] \arrow[d] & \tilde{Y} \arrow[r] \arrow[d] & \tilde{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{X}' \arrow[r] & \tilde{Y}' \arrow[r] & \tilde{Z}' \arrow[r] & 0 \end{tikzcd}$$

between exact sequences in $\mathcal{A}$ so that the canonical functor $\mathcal{A} \to \operatorname{St} \mathcal{A}$ maps the second to the first diagram.

Definition 9.1. Let $\mathcal{C}$ be a triangulated category and $\mathcal{X}$ a class of sequences in $\mathcal{C}$. We say that $\mathcal{X}$ is *phantomless* if for any pair of sequences X, Y in $\mathcal{X}$ we have

$$\lim_i{}^1 \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j) = 0.$$

This definition is consistent with our previous discussion of phantom morphisms in the following sense. Let $\mathcal{C} \subseteq \mathcal{T}$ be a triangulated subcategory consisting of compact objects such that $\mathcal{T}$ admits countable coproducts. Suppose that $\mathcal{X}$ is *stable under suspensions*, i.e. $(X_i)_{i \geq 0}$ in $\mathcal{X}$ implies $(\Sigma^n X_i)_{i \geq 0}$ in $\mathcal{X}$ for all $n \in \mathbb{Z}$. Then $\mathcal{X}$ is phantomless if and only if

$$\operatorname{Ph}(\operatorname*{hocolim}_i X_i, \operatorname*{hocolim}_j Y_j) = 0$$

for all X, Y in $\mathcal{X}$. This follows from Lemmas 8.3 and 8.4.

Proposition 9.2 ([13, Theorem 4.7])**.** *Let $\mathcal{C}$ be an algebraic triangulated category, viewed as a full subcategory of its sequential completion $\operatorname{Ind}_{\mathbb{N}} \mathcal{C}$. Let $\mathcal{X}$ be a class of sequences in $\mathcal{C}$ that is phantomless, closed under suspensions, and closed under the formation of cones. Then the full subcategory $\operatorname{Ind}_{\mathcal{X}} \mathcal{C} \subseteq \operatorname{Ind}_{\mathbb{N}} \mathcal{C}$ given by the colimits of sequences in $\mathcal{X}$ admits a unique triangulated structure such that the exact triangles are precisely the ones isomorphic to colimits of sequences that are given by coherent morphisms of exact triangles in $\mathcal{C}$.*

Let us spell out the triangulated structure for $\operatorname{Ind}_{\mathcal{X}} \mathcal{C}$. Fix a sequence of coherent morphisms $\eta_0 \to \eta_1 \to \eta_2 \to \cdots$ of exact triangles

$$\eta_i \colon X_i \longrightarrow Y_i \longrightarrow Z_i \longrightarrow \Sigma X_i$$

in $\mathcal{C}$ and suppose that it is also a sequence of morphisms $X \to Y \to Z \to \Sigma X$ of sequences in $\mathcal{X}$. This identifies with the sequence

$$\operatorname*{colim}_i X_i \longrightarrow \operatorname*{colim}_i Y_i \longrightarrow \operatorname*{colim}_i Z_i \longrightarrow \operatorname*{colim}_i \Sigma X_i$$

in $\operatorname{Ind}_{\mathcal{X}} \mathcal{C}$, and the exact triangles in $\operatorname{Ind}_{\mathcal{X}} \mathcal{C}$ are precisely sequences of morphisms that are isomorphic to sequences of the above form.

Proof of Proposition 9.2. We use the enhancement as follows. Suppose that $\mathcal{C} = \operatorname{St} \mathcal{A}$ for some Frobenius category $\mathcal{A}$. We denote by $\mathcal{C}^{\sim}$ the stable category $\operatorname{St} \mathcal{A}^{\sim}$ of the countable envelope of $\mathcal{A}$; see Example 2.4. This is a triangulated category with countable coproducts and $\mathcal{C}$ identifies with a full subcategory of compact objects.

Given sequences X, Y in $\mathcal{X}$ we set $\bar{X} = \operatorname{hocolim}_i X_i$ and $\bar{Y} = \operatorname{hocolim}_j Y_j$ in $\mathcal{C}^{\sim}$. Using that $\mathcal{X}$ is phantomless we compute

$$\lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j) \cong \operatorname{Hom}(h_X, h_Y) \cong \operatorname{Hom}(\bar{X}, \bar{Y})$$

where the first isomorphism follows from Lemma 8.3 and the second from Lemma 8.5. Thus taking a sequence in $\mathcal{X}$ to its homotopy colimit in $\mathcal{C}^{\sim}$ provides a fully faithful functor

$$\operatorname{hocolim} \colon \operatorname{Ind}_{\mathcal{X}} \mathcal{C} \longrightarrow \mathcal{C}^{\sim}.$$

Then it remains to compare the triangulated structures on both side, which turn out to be equivalent by construction. ☐

The above result admits a substantial generalisation, from algebraic triangulated categories to triangulated categories with a *morphic enhancement* in the sense of Keller [13, Appendix C]. Moreover, in some interesting cases the morphic enhancement extends to a morphic enhancement of the completion.

Example 9.3. For a right coherent ring A let $\mathcal{X}$ denote the class of Cauchy sequences $(X_i)_{i \geq 0}$ in $\mathbf{D}^b(\operatorname{proj} A)$ such that $\operatorname{colim}_i H^n(X_i) = 0$ for $|n| \gg 0$. Then $\mathcal{X}$ is phantomless and we have a triangle equivalence

$$\operatorname{Ind}_{\mathcal{X}} \mathbf{D}^b(\operatorname{proj} A) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} A).$$

We end these notes with a couple of references that complement our approach towards the completion of triangulated categories. The work of Neeman offers an intriguing approach that uses metrics on triangulated categories, thereby avoiding the use of any enhancements [19, 20]. On the other hand, there is Lurie's approach via stable ∞-categories [16, §1]; it uses a notion of enhancement that is far more sophisticated than the one presented in these notes.

References

[1] D. J. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, Local dualisable objects in local algebra, arXiv:2302.08562, 2023.

[2] D. J. Benson and H. Krause, Complexes of injective kG-modules, Algebra Number Theory **2** (2008), no. 1, 1–30.

[3] A. I. Bondal and M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[4] G. Cantor, Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen, Math. Ann. **5** (1872), no. 1, 123–132.

[5] W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra **22** (1994), no. 5, 1641–1674.

[6] W. G. Dwyer and J. P. C. Greenlees, Complete modules and torsion modules, Amer. J. Math. **124** (2002), no. 1, 199–220.

[7] J. P. C. Greenlees and J. P. May, Derived functors of I-adic completion and local homology, J. Algebra **149** (1992), no. 2, 438–453.

[8] A. Grothendieck and J. L. Verdier, Préfaisceaux, in *SGA 4, Théorie des Topos et Cohomologie Etale des Schémas, Tome 1. Théorie des Topos*, 1–184, Lecture Notes in Math., 269, Springer, Heidelberg, 1972.

[9] B. Keller, Chain complexes and stable categories, Manus. Math. **67** (1990), 379–417.

[10] H. Krause, Smashing subcategories and the telescope conjecture—an algebraic approach, Invent. Math. **139** (2000), no. 1, 99–133.

[11] H. Krause, Report on locally finite triangulated categories, J. K-Theory **9** (2012), no. 3, 421–458.

[12] H. Krause, Deriving Auslander's formula, Doc. Math. **20** (2015), 669–688.

[13] H. Krause, Completing perfect complexes, Math. Z. **296** (2020), no. 3-4, 1387–1427.

[14] H. Krause, *Homological theory of representations*, Cambridge Studies in Advanced Mathematics, 195, Cambridge University Press, Cambridge, 2022.

[15] H. Lenzing, Homological transfer from finitely presented to infinite modules, in *Abelian group theory (Honolulu, Hawaii, 1983)*, 734–761, Lecture Notes in Math., 1006, Springer, Berlin, 1983.

[16] J. Lurie, Higher Algebra, 2017.

[17] C. Méray, Remarques sur la nature des quantités définies par la condition de servir de limites à des variables données, Revue des Sociétés savantes, Sci. Math. phys. nat. (2) **4** (1869), 280–289.

[18] J. Milnor, On axiomatic homology theory, Pacific J. Math. **12** (1962), 337–341.

[19] A. Neeman, The categories $\mathcal{T}^c$ and $\mathcal{T}_c^b$ determine each other, arXiv:1806.064714, 2018.

[20] A. Neeman, Metrics on triangulated categories, J. Pure Appl. Algebra **224** (2020), no. 4, 106206, 13 pp.

[21] J. Xiao and B. Zhu, Locally finite triangulated categories, J. Algebra **290** (2005), no. 2, 473–490.

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
Email address: hkrause@math.uni-bielefeld.de