Koszul, Ringel, and Serre duality for strict polynomial functors

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Vorsicht Funktor!

"der Funktor" versus "das Funktor"

Strict polynomial functors

- Strict polynomial functors were introduced by Friedlander and Suslin [1997] in their work on the cohomology of finite group schemes.
- We use an equivalent description in terms of representations of divided powers, following expositions of Bousfield [1967], Kuhn [1998], and Pirashvili [2003].
- Divided powers were introduced by Eilenberg–MacLane [1954] and Cartan [1954/55] in their study of the (co)homology of spaces.

Koszul duality: work of Chałupnik and Touzé

- Classical Koszul duality relates the module categories of symmetric and exterior algebras.
- Koszul duality for strict polynomial functors is due to
  - Marcin Chałupnik [Advances in Math., 2008], and
  - Antoine Touzé [arXiv:1103.4580].

  Their work is motivated by calculations of functor cohomology.
- There is some earlier work in that direction (more than 20 years ago) by Akin, Buchsbaum, Donkin, Ringel ...

Koszul duality: various connections

The plan for this talk is ...

- to explain strict polynomial functors via representations of divided powers,
- to explain the Koszul duality à la Chałupnik and Touzé, using
  - a new monoidal structure for strict polynomial functors, and
  - the classical Koszul duality (for symmetric/exterior algebras),
- to explain the connection with Ringel duality for Schur algebras,
- to explain the connection with Serre duality.

Polynomial representations of $\Gamma = \mathrm{GL}_n(k)$

Fix an infinite field k and a positive integer n.

- A polynomial representation of Γ is a group homomorphism

$$\rho\colon \Gamma \longrightarrow \mathrm{GL}_N(k)$$

  such that for each $g \in \Gamma$ all entries of $\rho(g)$ are polynomials in the entries of g.
- A representation is homogeneous of degree d if all polynomials are homogeneous of degree d.

Theorem (Schur 1901, Green 1980)

- *Each polynomial representation decomposes into a direct sum of homogeneous representations.*
- *The degree d homogeneous polynomial representations of Γ are equivalent to modules over the Schur algebra $S_k(n, d)$.*

The setup

Here is the setup:

- k = a commutative ring (e.g. a field or $\mathbb{Z}$)
- P_k = the category of finitely generated projective k-modules
- $\mathfrak{S}_d$ = the symmetric group permuting d elements ($d \geq 0$)

Some operations on objects $V, W \in \mathsf{P}_k$:

- $V \otimes W$ = the tensor product over k
- $\mathrm{Hom}(V, W)$ = the group of k-linear maps $V \to W$
- $V^* = \mathrm{Hom}(V, k)$ = the k-dual
- $\mathfrak{S}_d$ acts on $V^{\otimes d} = \underbrace{V \otimes \cdots \otimes V}_{d}$

The category of divided powers

For $V \in \mathsf{P}_k$ and $d \geq 0$ define divided and symmetric powers:

- $\Gamma^d V = (V^{\otimes d})^{\mathfrak{S}_d} = \{x \in V^{\otimes d} \mid \sigma x = x \text{ for all } \sigma \in \mathfrak{S}_d\}$
- $S^d V = V^{\otimes d} / \langle \sigma x - x \mid x \in V^{\otimes d}, \sigma \in \mathfrak{S}^d \rangle$

Note:

- $\Gamma^d V, S^d V \in \mathsf{P}_k$ and $(\Gamma^d V)^* \cong S^d(V^*)$.
- The original definition of the divided powers $\Gamma^d V$ is different; but the symmetric tensors used here are isomorphic.

The category of divided powers $\Gamma^d \mathsf{P}_k$:

- objects of $\Gamma^d \mathsf{P}_k$ = objects of P_k (= f.g. projective k-modules)
- $\mathrm{Hom}_{\Gamma^d \mathsf{P}_k}(V, W) = \Gamma^d \mathrm{Hom}(V, W) \cong \mathrm{Hom}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$

Strict polynomial functors = $\operatorname{Rep}\Gamma_k^d$

Definition

A strict polynomial functor of degree d is a k-linear functor

$$X\colon \Gamma^d \mathsf{P}_k \longrightarrow \mathsf{M}_k = \text{category of } k\text{-modules}.$$

$\operatorname{Rep}\Gamma_k^d$ = category of degree d strict polynomial functors

- $X \in \operatorname{Rep}\Gamma_k^d$ consists of a pair of functions

$$V \mapsto X(V) \quad \text{and} \quad (V,W) \mapsto X_{V,W} \qquad (V, W \in \mathsf{P}_k)$$

$$X_{V,W}\colon \Gamma^d \operatorname{Hom}(V,W) \longrightarrow \operatorname{Hom}(X(V), X(W))$$

- Each k-linear map $X_{V,W}$ corresponds to a homogeneous polynomial map (or homogeneous polynomial law) of degree d

$$\operatorname{Hom}(V,W) \longrightarrow \operatorname{Hom}(X(V), X(W)).$$

Schur algebras

Let $V = k^n$. Then

$$\operatorname{End}_{\Gamma^d \mathsf{P}_k}(V) = \Gamma^d \operatorname{End}(V) \cong S_k(n,d)$$

where $S_k(n,d)$ denotes the Schur algebra [Green 1980].

Theorem (Friedlander–Suslin 1997)

Let $n \geq d$. The category $\operatorname{Rep} \Gamma^d_k$ is equivalent to the category of modules over $S_k(n,d)$ (via evaluation at k^n).

Corollary

Let k be an infinite field. Then

$$\operatorname{rep} \Gamma^d_k = \textit{category of } k\textit{-linear functors } \Gamma^d \mathsf{P}_k \to \mathsf{P}_k$$

is equivalent to the category of degree d homogeneous polynomial representations of $\mathrm{GL}_n(k)$.

Examples of strict polynomial functors

- The divided powers: $\Gamma^d\colon V \mapsto \Gamma^d V$
- The symmetric powers: $S^d\colon V \mapsto S^d V$
- The exterior powers: $\Lambda^d\colon V \mapsto \Lambda^d V$
- The dual of $X \in \operatorname{Rep}\Gamma^d_k$: $X^\circ\colon V \mapsto X(V^*)^*$
- The representable functors

$$\Gamma^{d,V} = \operatorname{Hom}_{\Gamma^d \mathsf{P}_k}(V, -) \qquad (V \in \mathsf{P}_k)$$

  form a set of projective generators, by Yoneda's lemma.
- Note: $S^d \cong (\Gamma^d)^\circ$ and $(\Lambda^d)^\circ \cong \Lambda^d$.

The external tensor product

Proposition

For integers $d, e \geq 0$ there is a bifunctor

$$- \otimes - \colon \operatorname{Rep}\Gamma_k^d \times \operatorname{Rep}\Gamma_k^e \longrightarrow \operatorname{Rep}\Gamma_k^{d+e}.$$

One defines $X \otimes Y \colon \Gamma^{d+e}\mathsf{P}_k \to \mathsf{M}_k$

- on objects via

$$(X \otimes Y)(V) = X(V) \otimes Y(V)$$

- on morphisms via

$$\Gamma^{d+e}\operatorname{Hom}(V, W) \subseteq \Gamma^d \operatorname{Hom}(V, W) \otimes \Gamma^e \operatorname{Hom}(V, W)$$

  which is induced by the inclusion

$$\mathfrak{S}_d \times \mathfrak{S}_e \subseteq \mathfrak{S}_{d+e}.$$

Decomposing divided powers

For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers set

$$\Gamma^\lambda = \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n} \quad \text{and} \quad S^\lambda = S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_n}.$$

Proposition

For integers $d, n \geq 0$ there is a canonical decomposition

$$\Gamma^{d,k^n} \cong \bigoplus_{\substack{\lambda=(\lambda_1,\ldots,\lambda_n) \\ \sum_i \lambda_i = d}} \Gamma^\lambda.$$

Corollary

- *Each Γ^λ is a projective object in $\operatorname{Rep}\Gamma^d_k$, where $d = \sum_i \lambda_i$.*
- *Each Γ^{d,k^n} is a projective generator of $\operatorname{Rep}\Gamma^d_k$, when $n \geq d$.*

The proof: how partitions come into play

- Consider the graded algebras of symmetric and divided powers

$$SV = \bigoplus_{i \geq 0} S^i V \quad \text{and} \quad \Gamma V = \bigoplus_{i \geq 0} \Gamma^i V \quad (V \in \mathsf{P}_k).$$

- The functor $V \mapsto SV$ preserves coproducts. Thus

$$SV \otimes SW \cong S(V \oplus W) \quad \text{and} \quad \Gamma V \otimes \Gamma W \cong \Gamma(V \oplus W).$$

- For each $n \geq 1$, this yields in $\operatorname{Rep} \Gamma^d_k$ a decomposition

$$\Gamma^{d,k^n} = \bigoplus_{i=0}^{d} (\Gamma^{d-i,k^{n-1}} \otimes \Gamma^i).$$

- Now use induction on n.

The internal tensor product

Proposition

For each $d \geq 0$ there are bifunctors

$$- \otimes_{\Gamma^d_k} - \colon \operatorname{Rep}\Gamma^d_k \times \operatorname{Rep}\Gamma^d_k \longrightarrow \operatorname{Rep}\Gamma^d_k$$

$$\mathcal{H}om_{\Gamma^d_k}(-,-) \colon (\operatorname{Rep}\Gamma^d_k)^{\mathrm{op}} \times \operatorname{Rep}\Gamma^d_k \longrightarrow \operatorname{Rep}\Gamma^d_k$$

It suffices to define these bifunctors on finitely generated projectives:

$$\Gamma^{d,V} \otimes_{\Gamma^d_k} \Gamma^{d,W} = \Gamma^{d,V\otimes W}$$

$$\mathcal{H}om_{\Gamma^d_k}(\Gamma^{d,V}, \Gamma^{d,W}) = \Gamma^{d,\operatorname{Hom}(V,W)}$$

Remark

- The construction proceeds in two steps: $\mathsf{P}_k \rightsquigarrow \Gamma^d\mathsf{P}_k \rightsquigarrow \operatorname{Rep}\Gamma^d_k$.
- The tensor product induces (via transport of structure) a tensor product for modules over the Schur algebras $S_k(n,d)$.

Koszul duality

Theorem

Let $d \geq 0$. Then

$$\Lambda^d \otimes_{\Gamma^d_k} \Lambda^d \cong S^d.$$

- There is also a derived version of this formula.
- The proof uses the classical Koszul duality.
- A consequence is the Koszul duality for $\operatorname{Rep}\Gamma^d_k$.

Koszul duality: derived version

$\mathrm{D}(\mathrm{Rep}\,\Gamma^d_k)$ = the derived category of $\mathrm{Rep}\,\Gamma^d_k$

Theorem

Let $d \geq 0$. Then

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \Lambda^d \cong S^d.$$

Corollary (Chałupnik, Touzé, K)

The functors $\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$ and $\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(\Lambda^d, -)$ provide mutually quasi-inverse equivalences

$$\mathrm{D}(\mathrm{Rep}\,\Gamma^d_k) \xrightarrow{\sim} \mathrm{D}(\mathrm{Rep}\,\Gamma^d_k).$$

Koszul duality: finite version

$\mathsf{D}^b(\operatorname{rep}\Gamma^d_k)$ = the bounded derived category of

$$\operatorname{rep}\Gamma^d_k = \text{ category of } k\text{-linear functors } \Gamma^d \mathsf{P}_k \to \mathsf{P}_k$$

Note: The natural functor $\mathsf{D}^b(\operatorname{rep}\Gamma^d_k) \to \mathsf{D}(\operatorname{Rep}\Gamma^d_k)$ is fully faithful.

Corollary

The functor $\mathbf{R}\mathcal{H}om_{\Gamma^d_k}(-,\Lambda^d)$ *induces an equivalence*

$$D\colon \mathsf{D}^b(\operatorname{rep}\Gamma^d_k)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{D}^b(\operatorname{rep}\Gamma^d_k)$$

satisfying $D^2 \cong \mathrm{Id}$.

The proof: resolutions of Λ^d and S^d

The strategy for the proof of

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} \Lambda^d \cong S^d :$$

- Construct a projective resolution of Λ^d in $\operatorname{Rep}\Gamma^d_k$.
- Apply $\Lambda^d \otimes_{\Gamma^d_k} -$ to this resolution and get a resolution of S^d.
- These resolutions are obtained from normalised bar resolutions, using classical Koszul duality [Totaro 1997].

Theorem (Classical Koszul duality)

For each $V \in \mathrm{P}_k$, there are isomorphisms of graded algebras:

$$\operatorname{Ext}_{S(V^*)}(k,k) \cong \Lambda V \quad \textit{and} \quad \operatorname{Ext}_{\Lambda(V^*)}(k,k) \cong SV$$

The proof: the projective resolution of Λ^d

The normalised bar resolution of k over $S(V^*)$:

$$\cdots \to S(V^*) \otimes S^{>0}(V^*)^{\otimes 2} \to S(V^*) \otimes S^{>0}(V^*) \to S(V^*) \to k \to 0$$

Apply $\mathrm{Hom}_{S(V^*)}(-, k)$:

$$0 \to k \longrightarrow S^{>0}(V^*)^* \to (S^{>0}(V^*)^*)^{\otimes 2} \to \cdots$$

The cohomology of this complex is ΛV. Taking the degree d part (with $S^i(V^*)^*$ replaced by $\Gamma^i V$) yields an exact sequence:

Resolution of Λ^d

$$0 \to \Gamma^d V \to \bigoplus_{i_1+i_2=d} \Gamma^{i_1} V \otimes \Gamma^{i_2} V \to \cdots$$

$$\to \bigoplus_{i_1+\cdots+i_{d-1}=d} \Gamma^{i_1} V \otimes \cdots \otimes \Gamma^{i_{d-1}} V \to V^{\otimes d} \to \Lambda^d V \to 0$$

The proof: the resolution of S^d

Analogously, the normalised bar resolution of k over $\Lambda(V^*)$ yields:

Resolution of S^d

$$0 \to \Lambda^d V \to \bigoplus_{i_1+i_2=d} \Lambda^{i_1} V \otimes \Lambda^{i_2} V \to \cdots$$
$$\to \bigoplus_{i_1+\cdots+i_{d-1}=d} \Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_{d-1}} V \to V^{\otimes d} \to S^d V \to 0$$

$\Lambda^d \otimes_{\Gamma_k^d} -$ maps the resolution of Λ^d to the resolution of S^d. We use:

Proposition

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ *with* $\sum_i \lambda_i = d$,

$$\Lambda^d \otimes_{\Gamma_k^d} \Gamma^\lambda \cong \Lambda^\lambda.$$

Example: Schur and Weyl functors

Akin, Buchsbaum and Weyman [1982] introduced Schur and Weyl functors, motivated by resolutions of determinantal ideals.

- $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition of weight $d = \sum_i \lambda_i$
- λ' = the conjugate partition of λ
- $\Gamma^\lambda = \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n} \in \operatorname{Rep} \Gamma^d_k$, analogously S^λ, Λ^λ

Define the Schur functor

$$S_\lambda = \text{ image of } \Lambda^{\lambda'} \xrightarrow{\Delta \otimes \cdots \otimes \Delta} \mathrm{Id}^{\otimes d} \xrightarrow{s_{\lambda'}} \mathrm{Id}^{\otimes d} \xrightarrow{\nabla \otimes \cdots \otimes \nabla} S^\lambda$$

and the Weyl functor

$$W_\lambda = \text{ image of } \Gamma^{\lambda} \xrightarrow{\Delta \otimes \cdots \otimes \Delta} \mathrm{Id}^{\otimes d} \xrightarrow{s_{\lambda}} \mathrm{Id}^{\otimes d} \xrightarrow{\nabla \otimes \cdots \otimes \nabla} \Lambda^{\lambda'}.$$

Theorem (Chałupnik 2008)

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} W_\lambda \cong S_{\lambda'}$$

Ringel duality

Recall: For $n \geq d$, evaluation at k^n induces an equivalence

$$\mathrm{Ev}_{k^n} \colon \operatorname{Rep}\Gamma_k^d \xrightarrow{\sim} \operatorname{Mod} S_k(n,d).$$

Note: The Schur algebra $S_k(n,d)$ is quasi-hereditary in the sense of Cline–Parshall–Scott.

Theorem (Ringel 1991)

A quasi-hereditary algebra A admits a characteristic tilting module T. The Ringel dual $A' = \mathrm{End}_A(T)$ is again quasi-hereditary and A'' is Morita equivalent to A.

Theorem (Donkin 1993)

$$S_k(n,d)' \cong S_k(n,d)$$

Koszul duality = Ringel duality

The characteristic tilting module for $S_k(n,d)$ is

$$T = \mathrm{Ev}_{k^n}(\Lambda^d \otimes_{\Gamma_k^d} \Gamma^{d,k^n})$$

and Koszul duality composed with evaluation at k^n gives

$$\phi\colon S_k(n,d) = \mathrm{End}_{\Gamma_k^d}(\Gamma^{d,k^n}) \xrightarrow{\sim} \mathrm{End}_{S_k(n,d)}(T).$$

Theorem

The following diagram commutes up to a natural isomorphism.

$$\begin{array}{ccccc}
\mathrm{D}(\mathrm{Rep}\,\Gamma_k^d) & & \xrightarrow[\sim]{\mathbf{R}\mathcal{H}om(\Lambda^d,-)} & & \mathrm{D}(\mathrm{Rep}\,\Gamma_k^d) \\
\downarrow \mathrm{Ev}_{k^n} \wr & & & & \wr\downarrow \mathrm{Ev}_{k^n} \\
\mathrm{D}(\mathrm{Mod}\,S_k(n,d)) & \xrightarrow[\sim]{\mathbf{R}\mathrm{Hom}(T,-)} & \mathrm{D}(\mathrm{Mod}\,\mathrm{End}(T)) & \xrightarrow[\sim]{\phi_*} & \mathrm{D}(\mathrm{Mod}\,S_k(n,d))
\end{array}$$

$(\text{Koszul duality})^2 = \text{Serre duality}$

Let k be a field. The category $\mathsf{D}^b(\operatorname{rep}\Gamma^d_k)$ is Hom-finite.

A Serre functor is an equivalence $F\colon \mathsf{D} \xrightarrow{\sim} \mathsf{D}$ with a nat. isomorphism

$$\operatorname{Hom}_{\mathsf{D}}(X,-)^* \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{D}}(-,FX)$$

for each $X \in \mathsf{D}$. This formalises the notion of Serre duality.

Theorem

Let k be a field. The functor

$$(\Lambda^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -)^2 \cong S^d \otimes^{\mathbf{L}}_{\Gamma^d_k} -$$

induces a Serre functor

$$\mathsf{D}^b(\operatorname{rep}\Gamma^d_k) \xrightarrow{\sim} \mathsf{D}^b(\operatorname{rep}\Gamma^d_k).$$

Summary / open problems

- The category $\operatorname{Rep}\Gamma_k^d$ of degree d strict polynomial functors admits a monoidal structure:

$$- \otimes_{\Gamma_k^d} - \quad \text{and} \quad \mathcal{H}om_{\Gamma_k^d}(-,-)$$

- Combining this monoidal structure with the classical Koszul duality yields the Koszul duality à la Chałupnik and Touzé:

$$\Lambda^d \otimes^{\mathbf{L}}_{\Gamma_k^d} -\colon \mathrm{D}(\operatorname{Rep}\Gamma_k^d) \xrightarrow{\sim} \mathrm{D}(\operatorname{Rep}\Gamma_k^d).$$

- What is the relation between these Koszul dualities?
- How can we compute the tensor product, say,

$$\Gamma^\lambda \otimes_{\Gamma_k^d} \Gamma^\mu \quad \text{or} \quad S_\lambda \otimes_{\Gamma_k^d} S_\mu$$

for partitions λ, μ of weigth d?