Support and cosupport of complexes

Henning Krause

Universität Bielefeld

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www.math.uni-bielefeld/~hkrause
Support and cosupport provide a link between

ALGEBRA AND GEOMETRY.

I discuss two papers of Amnon Neeman involving these concepts:


Applications in representation theory of finite groups follow at the end.

All this is part of a joint project with D. Benson and S. Iyengar.
Here is the setup:

- $R$ = a commutative noetherian ring
- $\text{Mod } R = \text{the category of } R\text{-modules}$
- $\mathbb{D}(R) = \text{the (unbounded) derived category of } \text{Mod } R$
- $\text{Spec } R = \text{the set of prime ideals of } R$

$\mathbb{D}(R)$ is a triangulated category with set-indexed (co)products.
Localizing and colocalizing subcategories

**Definition**

A triangulated subcategory $C \subseteq D(R)$ is called

- **localizing** if $C$ is closed under taking all coproducts,
- **colocalizing** if $C$ is closed under taking all products.

For any class $S \subseteq D(R)$ write:

- $\text{Loc}(S) =$ the smallest localizing subcategory containing $S$
- $\text{Coloc}(S) =$ the smallest colocalizing subcategory containing $S$
**Theorem (Neeman, 1992)**

The assignment

$$\text{Spec } R \supseteq \mathcal{U} \mapsto \text{Loc}(\{ k(p) \mid p \in \mathcal{U} \}) \subseteq D(R)$$

induces a bijection between

- the collection of **subsets** of Spec $R$, and
- the collection of **localizing subcategories** of $D(R)$.

Notation: $k(p) = \text{the residue field } R_p/p_p$
Classifying colocalizing subcategories

Theorem (Neeman, 2011)

The assignment

\[ \text{Spec } R \supseteq \mathcal{U} \mapsto \text{Coloc}(\{k(p) \mid p \in \mathcal{U}\}) \subseteq D(R) \]

induces a bijection between

\- the collection of subsets of Spec \( R \), and
\- the collection of colocalizing subcategories of \( D(R) \).

This is surprising because products tend to be complicated!

How are the results from '92 and '11 related to each other?

Is there a common proof?
For $C \subseteq D(R)$ write:

$$C^\perp = \{ X \in D(R) \mid \text{Hom}_{D(R)}(C, X) = 0 \text{ for all } C \in C \}$$

$$\perp C = \{ X \in D(R) \mid \text{Hom}_{D(R)}(X, C) = 0 \text{ for all } C \in C \}$$

- If $C$ is localizing, then $C^\perp$ is colocalizing.
- If $C$ is colocalizing, then $\perp C$ is localizing.
- If $C$ is localizing, then $\perp (C^\perp) = C$ [Neeman 1992].

**Corollary (Neeman, 2011)**

The assignment $C \mapsto C^\perp$ induces a bijection between

- the collection of localizing subcategories of $D(R)$, and
- the collection of colocalizing subcategories of $D(R)$. 
The support of a complex

**Definition (Foxby, 1979)**

For $X \in D(R)$ define the support

$$\text{supp } X = \{ p \in \text{Spec } R \mid X \otimes_R^L k(p) \neq 0 \}.$$

Some examples:
- If $X \in D^b(\text{mod } R)$, then
  $$\text{supp } X = \{ p \in \text{Spec } R \mid X_p \neq 0 \} = \bigcup_{n \in \mathbb{Z}} \text{supp } H^n(X).$$
- Let $p \in \text{Spec } R$. Then $\text{supp } E(R/p) = \text{supp } k(p) = \{ p \}.$

**Corollary (Neeman, 1992)**

For $X, Y \in D(R)$ we have

$$\text{supp } X \subseteq \text{supp } Y \iff \text{Loc}(X) \subseteq \text{Loc}(Y).$$
Let us test our understanding of localizing subcategories:

**Question**

Let $X \in D(R)$. Is it true that

$$\text{Loc}(X) = \text{Loc}(H^*(X))$$?
An example

\[ R = k[x, y] \text{ (} k \text{ a field)} \]
\[ \mathfrak{m} = (x, y) \text{ the maximal ideal of } R \]
\[ Q = \text{ the field of fractions of } R \]

The minimal injective resolution of \( R \):

\[
\cdots \rightarrow 0 \rightarrow Q \rightarrow \bigoplus_{\text{ht } p=1} E(R/p) \rightarrow E(R/m) \rightarrow 0 \rightarrow \cdots
\]

\[ X : \cdots \rightarrow 0 \rightarrow Q \rightarrow \bigoplus_{\text{ht } p=1} E(R/p) \rightarrow 0 \rightarrow \cdots \]

In \( D(R) \) one has

\[ \text{supp } X = (\text{Spec } R) \setminus \{ \mathfrak{m} \} \quad \text{supp } H^*(X) = \text{Spec } R \]

Thus \( \text{Loc}(X) \neq \text{Loc}(H^*X) \).
The cosupport of a complex

**Definition**

For $X \in D(R)$ define the cosupport

$$\text{cosupp } X = \{ p \in \text{Spec } R \mid R\text{Hom}_R(k(p), X) \neq 0 \}.$$

This seems hard to compute, even for ‘simple’ objects:

- Let $R = \mathbb{Z}$. Then $\text{cosupp } X = \text{supp } X$ for $X \in D^b(\text{mod } R)$.
- Let $(R, m)$ be complete local. Then $\text{cosupp } R = \{ m \}$.

**Proposition**

*For a complex $X$ in $D(R)$ we have*

$$\text{Max}(\text{supp } X) = \text{Max}(\text{cosupp } X).$$

**Notation:** $\text{Max } \mathcal{U} = \{ p \in \mathcal{U} \mid p \subseteq q \in \mathcal{U} \implies p = q \}$. 
Four fundamental (idempotent) functors $\text{Mod } R \to \text{Mod } R$:

- **localization** $M \mapsto M \otimes_R R_p$
- **colocalization** $\text{Hom}_R(R_p, M) \mapsto M$
- **torsion** $\Gamma_a M = \lim \rightarrow \text{Hom}(R/\mathfrak{a}^n, M) \mapsto M$
- **completion** $M \mapsto \Lambda_a M = \lim \leftarrow M \otimes_R R/\mathfrak{a}^n$

Their derived functors $D(R) \to D(R)$:

- **localization** $X \mapsto X \otimes_R^L R_p$
- **colocalization** $R\text{Hom}_R(R_p, X) \mapsto X$
- **local cohomology** $R\Gamma_a X \mapsto X$ [Grothendieck, 1967]
- **local homology** $X \mapsto L\Lambda_a X$ [Greenlees–May, 1992]

Note:

- The functor $R\text{Hom}_R(R_p, -)$ is a right adjoint of $- \otimes_R^L R_p$.
- The functor $L\Lambda_a$ is a right adjoint of $R\Gamma_a$. 
**Definition**

Fix \( p \in \text{Spec } R \) and define (by abuse of notation):

- **local cohomology** \( \Gamma_p = R\Gamma_p(- \otimes^L R_p) \),
- **local homology** \( \Lambda_p = R\text{Hom}_R(R_p, L\Lambda_p-) \).

These are idempotent functors \( D(R) \to D(R) \), and \( \Lambda_p \) is a right adjoint of \( \Gamma_p \).

We consider their essential images:

- \( \text{Im } \Gamma_p = \text{local cohomology objects} \) (a localizing subcategory)
- \( \text{Im } \Lambda_p = \text{local homology objects} \) (a colocalizing subcategory)

Note: \( \Lambda_p \) induces an equivalence \( \text{Im } \Gamma_p \sim \to \text{Im } \Lambda_p \).
An alternative description of (co)support:

- \( \text{supp } X = \{ p \in \text{Spec } R \mid \Gamma_p X \neq 0 \} \).
- \( \text{cosupp } X = \{ p \in \text{Spec } R \mid \Lambda_p X \neq 0 \} \).

The following are equivalent:

- \( H^n(X) \) is \( p \)-local and \( p \)-torsion for all \( n \in \mathbb{Z} \).
- \( \text{supp } X \subseteq \{ p \} \).
- \( X \) lies in \( \text{Im } \Gamma_p \).

There seems to be no analogue for \( \Lambda_p \).
**Proposition**

The assignment

\[ D(R) \supseteq C \leftrightarrow (C \cap \text{Im } \Gamma_p)_{p \in \text{Spec } R} \]

induces a bijection between

- the collection of localizing subcategories of \( D(R) \), and
- the collection of families \((C_p)_{p \in \text{Spec } R}\) with each \( C_p \subseteq \text{Im } \Gamma_p \) a localizing subcategory.

Analogously, the assignment

\[ D(R) \supseteq C \leftrightarrow (C \cap \text{Im } \Lambda_p)_{p \in \text{Spec } R} \]

classifies the colocalizing subcategories of \( D(R) \).
(Co)localizing subcategories of $\mathsf{D}(R)$

**Proposition**

Let $p \in \text{Spec } R$.

- $\text{Im } \Gamma_p$ has no proper localizing subcategories.
- $\text{Im } \Lambda_p$ has no proper colocalizing subcategories.

**Proof.**

For each $0 \neq X \in \text{Im } \Gamma_p$, one shows that

$$\text{Loc}(X) = \text{Loc}(k(p)) = \text{Im } \Gamma_p.$$

Analogously, $\text{Coloc}(Y) = \text{Im } \Lambda_p$ for each $0 \neq Y \in \text{Im } \Lambda_p$.

The classifications of [Neeman, 1992] and [Neeman, 2011] are immediate consequences.
Theorem

For $X, Y \in D(R)$ we have:

\[
\text{supp}(X \otimes^L_R Y) = (\text{supp } X) \cap (\text{supp } Y)
\]

\[
\text{cosupp}(\text{RHom}_R(X, Y)) = (\text{supp } X) \cap (\text{cosupp } Y)
\]
The above proof allows to generalize Neeman’s results to the derived category of a differential graded algebra $A$ such that
- $A$ is \textit{formal}, i.e. quasi-isomorphic to its cohomology $H^*(A)$,
- $H^*(A)$ is \textit{graded-commutative} and \textit{noetherian}.

An application to the study of \textit{modular representations of finite groups} goes as follows:

Let $G$ be a finite group and $k$ a field of characteristic $p > 0$. We consider modules over the group algebra $kG$ and classify the (co)localizing subcategories of the stable category $\text{StMod } kG$. 
Modular representations of finite groups

Take as example $G = (\mathbb{Z}/2\mathbb{Z})^r$ and a field $k$ of characteristic 2.

Group algebra $kG \cong k[x_1, \ldots, x_r]/(x_1^2, \ldots, x_r^2)$
Group cohomology $H^*(G, k) = \text{Ext}_{kG}^*(k, k) \cong k[\xi_1, \ldots, \xi_r]$

$K(\text{Inj} \, kG) = \text{category of complexes of injective } kG\text{-modules} / \text{htpy.}$
$ik = \text{an injective resolution of the trivial representation } k$
$\text{End}_{kG}(ik) = \text{the endomorphism dg algebra of } ik \text{ (is formal)}$

$$\text{StMod } kG \xrightarrow{\sim} K_{ac}(\text{Inj} \, kG) \hookrightarrow K(\text{Inj} \, kG)$$
$$\xrightarrow{\sim} \text{D(End}_{kG}(ik)) \xrightarrow{\sim} \text{D}(k[\xi_1, \ldots, \xi_r])$$

**Corollary**

There are canonical bijections between

- (co)localizing subcategories of $\text{StMod } kG$, and
- sets of graded non-maximal prime ideals of $H^*(G, k)$. 
The next lecture will focus on infinite methods:

- Compactly generated triangulated categories
- Bousfield localization
- Brown representability

Then we explain the stratification of triangulated categories using:

- Local cohomology functors
- Support of objects
- Local-global principle
My collaborators

Dave Benson and Srikanth Iyengar
(at an Oberwolfach Seminar 2010)