Functors and morphisms determined by objects were introduced in 1978 by Maurice Auslander in his celebrated Philadelphia notes.

The review begins: *This extremely long paper (244 pages) is devoted to the investigation of functors and morphisms determined by objects.*

The review ends: *The paper is clearly and concisely written. However, in view of the length of the paper, a table of contents would have been very useful.*
Fix a category, for example a module category.
Can we classify all morphisms ending in a fixed object?

Two morphisms $\alpha_i : X_i \to Y$ ($i = 1, 2$) are isomorphic if there exists an isomorphism $\phi : X_1 \to X_2$ such that $\alpha_1 = \alpha_2 \phi$. 

\[
\begin{array}{c}
X_1 \xrightarrow{\alpha_1} Y \\
\downarrow \phi \\
X \xrightarrow{\alpha_2} Y
\end{array}
\]
**Morphisms determined by objects**

**Definition (Auslander)**

Fix a category $C$. A morphism $\alpha : X \to Y$ in $C$ is right determined by a an object $C$ if for every morphism $\alpha' : X' \to Y$ the following are equivalent:

- $\alpha'$ factors through $\alpha$;
- for every $\phi : C \to X'$ the composite $\alpha' \phi$ factors through $\alpha$.

\[
\begin{array}{c}
C \xrightarrow{\phi} X' \xrightarrow{\alpha'} Y \\
\downarrow \quad \downarrow \quad \downarrow \\
\quad X \xrightarrow{\alpha} Y
\end{array}
\]

$\text{Im } \text{Hom}(C, \alpha) := \text{image of } \text{Hom}(C, \alpha) : \text{Hom}(C, X) \to \text{Hom}(C, Y)$

The second condition means

$\text{Im } \text{Hom}(C, \alpha') \subseteq \text{Im } \text{Hom}(C, \alpha)$. 
**Example: almost split morphisms**

**Definition (Auslander–Reiten)**

A morphism \( \alpha : X \to Y \) is right almost split if \( \alpha \) is not a retraction and every morphism \( X' \to Y \) that is not a retraction factors through \( \alpha \).

**Proposition (Auslander)**

A morphism \( \alpha : X \to Y \) in an additive category is right almost split if and only if

- \( \text{End}(Y) \) is a local ring,
- \( \alpha \) is right determined by \( Y \),
- \( \text{Im Hom}(Y, \alpha) = \text{rad End}(Y) \).
A morphism $\alpha: X \to Y$ often decomposes as follows: There is a decomposition

$$X = X' \oplus X''$$

such that $\alpha|_{X'}$ is right minimal and $\alpha|_{X''} = 0$.

**Observation**

Let $\alpha_i: X_i \to Y$ ($i = 1, 2$) be morphisms that are right minimal and right $C$-determined. Then

$$\alpha_1 \cong \alpha_2 \iff \text{Im } \text{Hom}(C, \alpha_1) = \text{Im } \text{Hom}(C, \alpha_2).$$
Fix a ring $\Lambda$ and consider the category $\text{Mod} \Lambda$ of $\Lambda$-modules.

**Theorem (Auslander)**

Let $C$ and $Y$ be $\Lambda$-modules with $C$ finitely presented. Given an $\text{End}_\Lambda(C)$-submodule $H \subseteq \text{Hom}_\Lambda(C, Y)$, there exists a right $C$-determined morphism $\alpha: X \to Y$ in $\text{Mod} \Lambda$ satisfying

$$\text{Im } \text{Hom}_\Lambda(C, \alpha) = H.$$

**Theorem (Auslander)**

Let $\Lambda$ be an Artin algebra. A morphism $\alpha: X \to Y$ in $\text{mod} \Lambda$ is right determined as a morphism in $\text{Mod} \Lambda$ by

$$\text{Tr } D(\text{Ker } \alpha) \oplus P(\text{Coker } \alpha).$$
Auslander’s work: some comments

- Auslander’s formulation of the first result includes a uniqueness statement: $\alpha$ can be chosen to be right minimal.
- The proofs are based on the Auslander–Reiten formula

$$D\text{Hom}(X, Y) \cong \text{Ext}^1(Y, D \text{Tr} X).$$

- The first result is a vast generalisation of the existence result for almost split sequences:
- Take for $Y$ a finitely presented module with local endomorphism ring, $C = Y$, and $H = \text{rad End}(Y)$. Then $\alpha: X \to Y$ is right almost split.
- The results are not restricted to the category of finitely presented modules; their proofs involve pure-injective modules.
Motivated by Auslander’s results, we divide the classification problem into two parts:

**Question**

Fix a category $C$ and an object $Y \in C$.

- **Morphisms:** Given an object $C \in C$ and a subset $H \subseteq \text{Hom}(C, Y)$ which is $\text{End}(C)$-invariant. Is there a right $C$-determined morphism $\alpha : X \to Y$ with $\text{Im} \text{Hom}(C, \alpha) = H$?

- **Objects:** Is every morphism ending in $Y$ right determined by an object?
Let $C$ be a partially ordered set, viewed as a category.

- Fix a morphism $\alpha: x \to y$, which means that $x \leq y$.
- If $x = y$, then $\alpha$ is right determined by every object of $C$.
- If $x \neq y$, then $\alpha$ is right determined by an object $c \in C$ iff
  \[ C_\alpha = \{ c \in C \mid c \not\leq x, c \leq y \} \]
  has a unique minimal element. In that case $c = \inf C_\alpha$.

- In $(\mathbb{Z}, \leq)$, all morphisms are determined by objects.
- In $(\mathbb{Q}, \leq)$, only identity morphisms are determined by objects.
**Definition (Reiten–Van den Bergh)**

Let $k$ be a field and $C$ a $k$-linear additive category with finite dimensional Hom-spaces. A right Serre functor is an additive functor $S: C \to C$ together with a natural isomorphism

$$D \text{Hom}(X, -) \xrightarrow{\sim} \text{Hom}(-, SX)$$

for all $X \in C$, where $D = \text{Hom}_k(-, k)$. A right Serre functor is called Serre functor if it is an equivalence.
Theorem

For a Hom-finite triangulated category $C$ are equivalent:

1. There is a right Serre functor $C \to C$.
2. Given objects $C$, $Y$ in $C$ and an $\text{End}(C)$-submodule $H \subseteq \text{Hom}(C, Y)$, there exists a morphism $\alpha : X \to Y$ which is right $C$-determined and satisfies $\text{Im} \text{Hom}(C, \alpha) = H$.

Note: A right Serre functor is unique up to isomorphism.

Theorem

For a right Serre functor $S : C \to C$ are equivalent:

1. The functor $S$ is a Serre functor.
2. Every morphism in $C$ is right determined by an object in $C$.

In this case a morphism with cone $C$ is right determined by $S^{-1}C$. 

We thank him: