Functors and Morphisms Determined by Objects Revisited

Henning Krause

Universität Bielefeld

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www.math.uni-bielefeld/~hkrause

A bit of history

- Functors and morphisms determined by objects were introduced in 1978 by Maurice Auslander in his celebrated Philadelphia notes.
- The review begins: *This extremely long paper (244 pages) is devoted to the investigation of functors and morphisms determined by objects.*
- The review ends: *The paper is clearly and concisely written. However, in view of the length of the paper, a table of contents would have been very useful.*

Motivating problem

- Fix a category, for example a module category.
- Can we classify all morphisms ending in a fixed object?
- Two morphisms $\alpha_i \colon X_i \to Y$ $(i = 1, 2)$ are isomorphic if there exists an isomorphism $\phi \colon X_1 \to X_2$ such that $\alpha_1 = \alpha_2 \phi$.

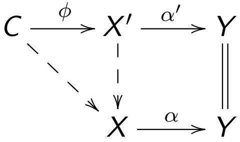
$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha_1} & Y \\ {\scriptstyle\phi}\downarrow & & \| \\ X & \xrightarrow{\alpha_2} & Y \end{array}$$

Morphisms determined by objects

Definition (Auslander)

Fix a category C. A morphism $\alpha\colon X \to Y$ in C is right determined by a an object C if for every morphism $\alpha'\colon X' \to Y$ the following are equivalent:

- α' factors through α;
- for every $\phi\colon C \to X'$ the composite $\alpha'\phi$ factors through α.



$\operatorname{Im}\operatorname{Hom}(C,\alpha) :=$ image of $\operatorname{Hom}(C,\alpha)\colon \operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$
The second condition means

$$\operatorname{Im}\operatorname{Hom}(C,\alpha') \subseteq \operatorname{Im}\operatorname{Hom}(C,\alpha).$$

Example: almost split morphisms

Definition (Auslander–Reiten)

A morphism $\alpha\colon X \to Y$ is right almost split if α is not a retraction and every morphism $X' \to Y$ that is not a retraction factors through α.

Proposition (Auslander)

A morphism $\alpha\colon X \to Y$ in an additive category is right almost split if and only if

- $\operatorname{End}(Y)$ *is a local ring,*
- α *is right determined by* Y*,*
- $\operatorname{Im}\operatorname{Hom}(Y, \alpha) = \operatorname{rad}\operatorname{End}(Y)$.

The classication problem: invariants

A morphism $\alpha\colon X \to Y$ often decomposes as follows: There is a decomposition

$$X = X' \oplus X''$$

such that $\alpha|_{X'}$ is right minimal and $\alpha|_{X''} = 0$.

Observation

Let $\alpha_i\colon X_i \to Y$ $(i = 1, 2)$ be morphisms that are right minimal and right C-determined. Then

$$\alpha_1 \cong \alpha_2 \quad \Longleftrightarrow \quad \operatorname{Im}\operatorname{Hom}(C, \alpha_1) = \operatorname{Im}\operatorname{Hom}(C, \alpha_2).$$

Auslander's work

Fix a ring Λ and consider the category $\operatorname{Mod}\Lambda$ of Λ-modules.

Theorem (Auslander)

Let C and Y be Λ-modules with C finitely presented. Given an $\operatorname{End}_\Lambda(C)$-submodule $H \subseteq \operatorname{Hom}_\Lambda(C, Y)$, there exists a right C-determined morphism $\alpha\colon X \to Y$ in $\operatorname{Mod}\Lambda$ satisfying

$$\operatorname{Im}\operatorname{Hom}_\Lambda(C, \alpha) = H.$$

Theorem (Auslander)

Let Λ be an Artin algebra. A morphism $\alpha\colon X \to Y$ in $\operatorname{mod}\Lambda$ is right determined as a morphism in $\operatorname{Mod}\Lambda$ by

$$\operatorname{Tr} D(\operatorname{Ker}\alpha) \oplus P(\operatorname{Coker}\alpha).$$

Auslander's work: some comments

- Auslander's formulation of the first result includes a uniqueness statement: α can be chosen to be right minimal.
- The proofs are based on the Auslander–Reiten formula

$$D\underline{\mathrm{Hom}}(X, Y) \cong \mathrm{Ext}^1(Y, D\,\mathrm{Tr}\,X).$$

- The first result is a vast generalisation of the existence result for almost split sequences:
- Take for Y a finitely presented module with local endomorphism ring, $C = Y$, and $H = \mathrm{rad}\,\mathrm{End}(Y)$. Then $\alpha\colon X \to Y$ is right almost split.
- The results are not restricted to the category of finitely presented modules; their proofs involve pure-injective modules.

The classification problem: reformulation

Motivated by Auslander's results, we divide the classification problem into two parts:

Question

Fix a category C and an object $Y \in \mathsf{C}$.

- Morphisms: Given an object $C \in \mathsf{C}$ and a subset $H \subseteq \operatorname{Hom}(C, Y)$ which is $\operatorname{End}(C)$-invariant. Is there a right C-determined morphism $\alpha\colon X \to Y$ with $\operatorname{Im} \operatorname{Hom}(C, \alpha) = H$?
- Objects: Is every morphism ending in Y right determined by an object?

Example: partially ordered sets

Example

Let C be a partially ordered set, viewed as a category.

- Fix a morphism $\alpha\colon x \to y$, which means that $x \leq y$.
- If $x = y$, then α is right determined by every object of C.
- If $x \neq y$, then α is right determined by an object $c \in \mathsf{C}$ iff

$$\mathsf{C}_\alpha = \{c \in \mathsf{C} \mid c \not\leq x, c \leq y\}$$

has a unique minimal element. In that case $c = \inf \mathsf{C}_\alpha$.

- In $(\mathbb{Z}, \leq)$, all morphisms are determined by objects.
- In $(\mathbb{Q}, \leq)$, only identity morphisms are determined by objects.

Serre functors: definition

Definition (Reiten–Van den Bergh)

Let k be a field and C a k-linear additive category with finite dimensional Hom-spaces. A right Serre functor is an additive functor $S\colon \mathsf{C} \to \mathsf{C}$ together with a natural isomorphism

$$D\operatorname{Hom}(X,-) \xrightarrow{\sim} \operatorname{Hom}(-,SX)$$

for all $X \in \mathsf{C}$, where $D = \operatorname{Hom}_k(-,k)$. A right Serre functor is called Serre functor if it is an equivalence.

Serre functors for triangulated categories

Theorem

For a Hom-finite triangulated category C *are equivalent:*

- *There is a right Serre functor* $\mathsf{C} \to \mathsf{C}$.
- *Given objects* C, Y *in* C *and an* $\operatorname{End}(C)$*-submodule* $H \subseteq \operatorname{Hom}(C, Y)$, *there exists a morphism* $\alpha\colon X \to Y$ *which is right* C*-determined and satisfies* $\operatorname{Im}\operatorname{Hom}(C, \alpha) = H$.

Note: A right Serre functor is unique up to isomorphism.

Theorem

For a right Serre functor $S\colon \mathsf{C} \to \mathsf{C}$ *are equivalent:*

- *The functor* S *is a Serre functor.*
- *Every morphism in* C *is right determined by an object in* C.

In this case a morphism with cone C *is right determined by* $S^{-1}C$.

We thank him: