Morphisms determined by objects and flat covers

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Motivating problem

Problem

- *Fix an additive category, for example a module category.*
- *Is there a procedure for constructing morphisms ending at a fixed object?*

More precisely, we are looking for:

- Invariants of morphisms ending at some fixed object.
- Constructions for universal morphisms with respect to these invariants.

Our strategy

We combine two concepts:

- Functors/morphisms determined by objects [Auslander, 1978]
- The existence of flat covers [Bican–El Bashir–Enochs, 2001]

Our Setting

We fix:

- A = an additive category, for example a module category.
- C = a set of objects of A, viewed as full subcategory.

Definition

- A C-module is an additive functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$.
- The category of C-modules is denoted by $(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab})$.

Example

$\mathsf{C} = \{C\}$ with $\Gamma := \mathsf{End}_{\mathsf{A}}(C)$. Then $(\mathsf{C}^{\mathrm{op}}, \mathsf{Ab}) = \mathsf{Mod}\,\Gamma$.

The restricted Yoneda functor

The pair of categories A and C gives a functor

$$\mathsf{A} \longrightarrow (\mathsf{C}^{\mathrm{op}}, \mathsf{Ab}), \quad X \mapsto \mathrm{Hom}_{\mathsf{A}}(\mathsf{C}, X)$$

by setting

$$\mathrm{Hom}_{\mathsf{A}}(\mathsf{C}, X) := \mathrm{Hom}_{\mathsf{A}}(-, X)|_{\mathsf{C}}.$$

Invariants of morphisms

For a morphisms $\alpha\colon X \to Y$ in A consider the image

$$\operatorname{Im} \operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, \alpha) \subseteq \operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, Y)$$

of the induced morphism

$$\operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, X) \longrightarrow \operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, Y) \quad \text{in} \quad (\mathsf{C}^{\mathrm{op}}, \mathsf{Ab}).$$

An existence result

Theorem

Suppose we have:

- A = *a locally finitely presented additive category.*
- C = *a set of finitely presented objects.*
- $Y \in \mathsf{A}$ *an object and* $H \subseteq \operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, Y)$ *a submodule.*

There exists an (essentially unique) morphism $\alpha\colon X \to Y$ *in* A *such that:*

- $\operatorname{Im}\operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, \alpha) = H$ *and any morphism* $\alpha'\colon X' \to Y$ *with* $\operatorname{Im}\operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, \alpha') \subseteq H$ *factors through* α.
- α *is right minimal (i.e. any* $\phi \in \operatorname{End}_{\mathsf{A}}(X)$ *with* $\alpha\phi = \alpha$ *is invertible).*

Morphisms determined by objects

Definition (Auslander)

A morphisms $\alpha\colon X \to Y$ in A is right C-determined if for every morphisms $\alpha'\colon X' \to Y$

$$\operatorname{Im}\operatorname{Hom}_{\mathsf{A}}(\mathsf{C},\alpha') \subseteq \operatorname{Im}\operatorname{Hom}_{\mathsf{A}}(\mathsf{C},\alpha) \implies \alpha' \text{ factors through } \alpha$$

For $\mathsf{A} = \operatorname{Mod}\Lambda$ and $\mathsf{C} = \{C\}$, the theorem is due to Auslander.

Example: almost split morphisms

Definition (Auslander–Reiten)

A morphism $\alpha \colon X \to Y$ is right almost split if

- α is not a retraction, and
- $\alpha' \colon X' \to Y$ not a retraction implies α' factors through α.

Proposition (Auslander)

A morphism $\alpha \colon X \to Y$ in an additive category is right almost split if and only if

- $\operatorname{End}(Y)$ *is a local ring,*
- α *is right determined by* Y*,*
- $\operatorname{Im} \operatorname{Hom}(Y, \alpha) = \operatorname{rad} \operatorname{End}(Y)$.

Existence of almost split split morphisms

Corollary

Let Y be a finitely presented object in a locally finitely presented additive category such that $\mathrm{End}(Y)$ *is local. Then there exists a (right minimal) right almost split morphism* $X \to Y$.

For module categories, this is due to Auslander.

Almost split sequences

In a module category, a right minimal and right almost split morphism $\alpha\colon X \to Y$ induces an almost split sequence

$$0 \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow X \longrightarrow Y \longrightarrow 0$$

provided Y is not projective.

Example

In $\mathrm{Mod}\,\mathbb{Z}$ there is no almost split sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Q} \longrightarrow 0.$$

Locally finitely presented categories

Definition (Crawley-Boevey)

An addtive category A is locally finitely presented if

- A has direct limits (= filtered colimits),
- the isoclasses of finitely presented objects in A form a set,
- every object is a direct limit of finitely presented objects.

$X \in \mathsf{A}$ is finitely presented if $\mathrm{Hom}_{\mathsf{A}}(X, -)$ preserves direct limits.

Example

Let Λ be a ring.

- $\mathrm{Mod}\,\Lambda$ = the category of Λ-modules
- $\mathrm{Flat}\,\Lambda$ = the category of flat Λ-modules

When is a morphisms determined by objects?

Proposition

For a morphism $\alpha\colon X \to Y$ are equivalent:

- *α is determined by a set of finitely presented objects.*
- *There is a decomposition $X = X' \oplus X''$ such that $\mathrm{Ker}(\alpha|_{X'})$ is pure injective and $\alpha|_{X''} = 0$.*

A bit of history

- Functors and morphisms determined by objects were introduced in 1978 by Maurice Auslander in his celebrated Philadelphia notes.
- The review begins: *This extremely long paper (244 pages) is devoted to the investigation of functors and morphisms determined by objects.*
- The review ends: *The paper is clearly and concisely written. However, in view of the length of the paper, a table of contents would have been very useful.*
- Auslander himself was very passionate about this work, but ...

An open problem

Problem (Auslander)

Describe the right C*-determined morphism* $\alpha\colon X \to Y$ *with* $\operatorname{Im}\operatorname{Hom}_{\mathsf{A}}(\mathsf{C}, \alpha) = 0$.

Note: $\alpha = 0$ when C contains a generator of A.

The Auslander bijection

For a category A, the morphisms ending at $Y \in \mathsf{A}$ are pre-ordered:

$$\alpha' \leq \alpha \quad :\Longleftrightarrow \quad \alpha' \text{ factors through } \alpha$$

Write $[\mathsf{A}/Y]$ for this poset (after identifying $\alpha' = \alpha$ when $\alpha' \leq \alpha$ and $\alpha \leq \alpha'$).

Theorem (Ringel)

Let Λ be an Artin algebra. For $Y \in \operatorname{mod} \Lambda$ there is an isomorphism

$$\operatorname*{colim}_{C \in \operatorname{mod} \Lambda} \operatorname{sub}(\operatorname{Hom}_\Lambda(C, Y)) \xrightarrow{\sim} [\operatorname{mod} \Lambda / Y]$$

given by the assignment

$$\operatorname{Hom}_\Lambda(C, Y) \supseteq H \longmapsto \alpha_{C,H}.$$

Flat covers

The proof of the existence result (for C-determined morphisms) is based on:

Theorem (Bican–El Bashir–Enochs, 2001)

Every additive functor admits a flat cover.

A flat cover is the analogue of a projective cover (replacing the term ‘projective’ by ‘flat’).

Next we explain:

Theorem

Flat covers are projective covers.

Finitely presented functors

For an additive category A we write

$$\mathsf{Fp}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$$

for the category of functors $F\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ having a presentation

$$\mathsf{Hom}_{\mathsf{A}}(-, X) \longrightarrow \mathsf{Hom}_{\mathsf{A}}(-, Y) \longrightarrow F \longrightarrow 0.$$

Proposition

Let A *be locally finitely presented. Then* $\mathsf{Fp}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$ *is an abelian category and*

$$\mathsf{A} \longrightarrow \mathsf{Fp}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab}), \quad X \mapsto \mathsf{Hom}_{\mathsf{A}}(-, X)$$

identifies A *with the full subcategory of projective objects.*

Flat covers are projective covers

Theorem

Let A *be a locally finitely presented category.*

- *For an additive functor* $F\colon (\operatorname{fp}\mathsf{A})^{\mathrm{op}} \to \mathsf{Ab}$*, the unique functor* $\widetilde{F}\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ *extending* F *and preserving filtered colimits in* A *is finitely presented and admits a minimal projective presentation in* $\operatorname{Fp}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab})$*.*
- *The assignment* $F \mapsto \widetilde{F}$ *provides a fully faithful right adjoint to the functor*

$$\operatorname{Fp}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab}) \longrightarrow ((\operatorname{fp}\mathsf{A})^{\mathrm{op}}, \mathsf{Ab}), \quad F \mapsto F|_{\operatorname{fp}\mathsf{A}}.$$

Example: $\mathsf{A} = \operatorname{Flat}\Lambda$

Let Λ be a ring and $\mathsf{A} = \operatorname{Flat}\Lambda$. Then

$$\operatorname{fp}\mathsf{A} = \operatorname{proj}\Lambda \quad \text{and} \quad ((\operatorname{fp}\mathsf{A})^{\mathrm{op}}, \mathsf{Ab}) \xrightarrow{\sim} \operatorname{Mod}\Lambda$$

For $Y \in \operatorname{Mod}\Lambda$ the theorem yields a projective cover

$$\operatorname{Hom}_{\mathsf{A}}(-, X) \longrightarrow \widetilde{Y} \quad \text{in} \quad \operatorname{Fp}(\mathsf{A}^{\mathrm{op}}, \mathsf{Ab}).$$

Evaluation at Λ then gives a flat cover

$$X = \operatorname{Hom}_{\mathsf{A}}(\Lambda, X) \longrightarrow \widetilde{Y}(\Lambda) = Y.$$

We thank him:

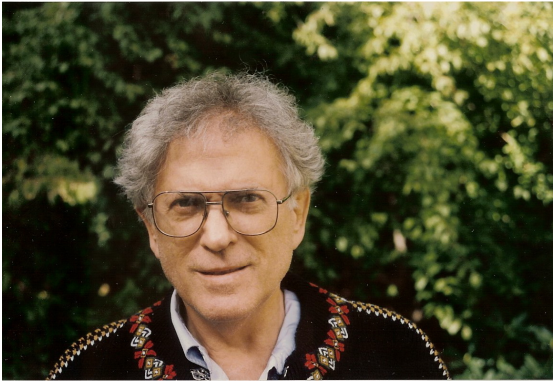

Photo: Gordana Todorov