THE TAMARI LATTICE IN REPRESENTATION THEORY

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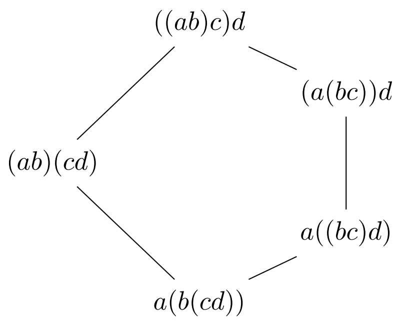
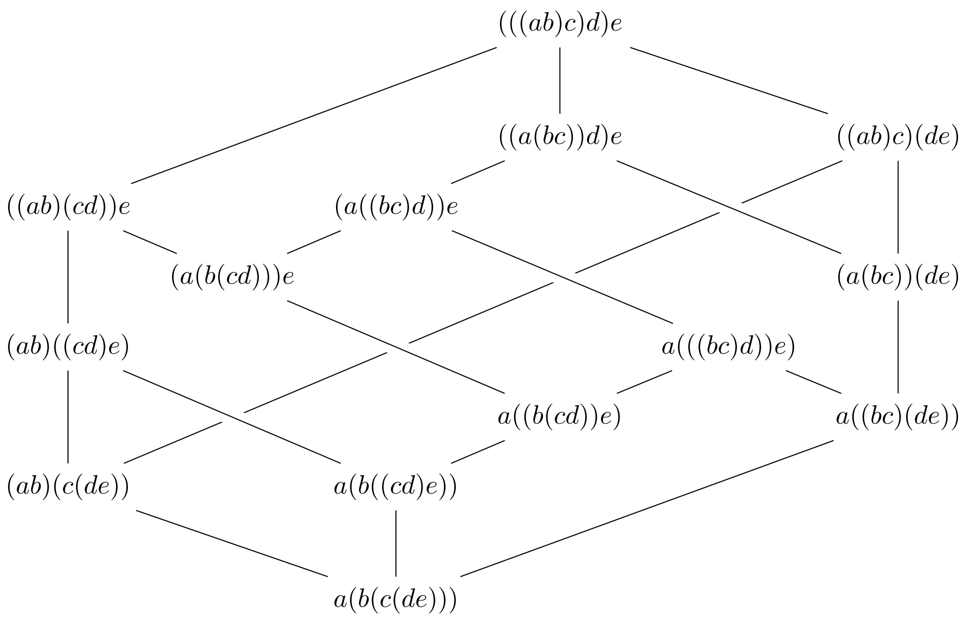
The Tamari lattice. Fix an integer $n \geq 1$. The *Tamari lattice* of order n is a partially ordered set and denoted by T_n. The elements consist of the meaningful bracketings of a string of $n+1$ letters. The partial order is given by applying the rule $(xy)z \to x(yz)$ from left to right. For example, when $n = 3$, we have

$$((ab)c)d \geq (a(bc))d \geq a((bc)d) \geq a(b(cd)).$$

Here is the Hasse diagram of the lattice T_3:



And here is the Hasse diagram of the lattice T_4:



The cardinality of the Tamari lattice T_n equals the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Let $\mathcal{I}(n)$ denote the set of intervals $[i, j] = \{i, i+1, \ldots, j\}$ in $\mathbb{Z}$ with $0 \leq i < j \leq n$. Two intervals I, J are said to be *compatible* if $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$. We write $\mathcal{C}(n)$ for the set of all subsets $X \subseteq \mathcal{I}(n)$ of cardinality n such that all intervals in X are pairwise compatible.

Lemma 1. *Sending an interval $[i, j]$ to the bracketing $x_0 \dots (x_i \dots x_j) \dots x_n$ of the string $x_0 \dots x_n$ induces a bijection $\mathcal{C}(n) \xrightarrow{\sim} T_n$.* □

Representations. Fix an integer $n \geq 1$ and a field k. We consider the quiver of type A_n with linear orientation

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

and denote by Λ_n its path algebra over k. For each $j \in \{1, \dots, n\}$ let P_j denote the indecomposable projective Λ_n-module having as a k-basis all paths ending in the vertex j, and for each interval $[i, j]$ in $\mathbb{Z}$ with $0 \leq i < j \leq n$ we set $M_{[i,j]} := P_j / \operatorname{rad}^{j-i} P_j$.

Lemma 2.
(1) *The set $\{M_I \mid I \in \mathcal{I}(n)\}$ is a complete set of isomorphism classes of indecomposable Λ_n-modules.*
(2) $\operatorname{Ext}^1(M_I, M_J) = 0 = \operatorname{Ext}^1(M_J, M_I)$ *if and only if the intervals I and J are compatible.*
(3) *There is an epimorphism $M_I \to M_J$ if and only if* $\sup I = \sup J$ *and* $\operatorname{card} I \geq \operatorname{card} J$. □

A Λ_n-module T is a *basic tilting module* if T has precisely n pairwise non-isomorphic indecomposable direct summands and $\operatorname{Ext}^1(T, T) = 0$. We write $T \geq T'$ if there is an epimorphism $T^r \to T'$ for some positive integer r. This induces a partial order on the isomorphism classes of basic tilting modules.

Proposition 3. *The assignment $X \mapsto \bigoplus_{I \in X} M_I$ induces a bijection between $\mathcal{C}(n)$ and the set of isomorphism classes of basic tilting modules over Λ_n. Composition with the bijection $T_n \xrightarrow{\sim} \mathcal{C}(n)$ yields a lattice isomorphism.* □

Historical comments. Eugène Charles Catalan noticed that the Catalan number C_n counts the bracketings of a string of $n + 1$ letters [2, 5]. The partial order on the set of bracketings was introduced by Dov Tamari [7]. This is part of a higher structure: the Hasse diagram of the Tamari lattice T_n is precisely the 1-skeleton of the $(n - 1)$-dimensional *Stasheff associahedron* [6, 4]. In the context of representations of quivers, Gabriel noticed that the Catalan number C_n counts the tilting modules of the equioriented quiver of type A_n [3]. The connection with the Tamari lattice (Proposition 3) was pointed out by Buan and Krause [1, 8].

References

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