Stratification of modular representations of finite groups and dualisable objects

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Gallia est omnis divisa in partes tres ...[1]

The whole of this talk is divided into three parts:

- Stratification of modular representations via cohomology (Benson–Iyengar–K, 2011)
- Stratification of triangulated categories – more abstractly (speculative)
- Dualisable objects and local regularity (Benson–Iyengar–K–Pevtsova, 2024)

[1] J. Caesar, De bello Gallico

Part I: Modular representations

Set-up

- G a finite group
- k a field
- kG the group algebra
- $\operatorname{Mod} kG$ the category of kG-modules
- $H^*(G, k) = \operatorname{Ext}^*_{kG}(k, k)$ the group cohomology

Proposition

The category of kG-modules is a Frobenius category and is endowed with a tensor product (over k with diagonal G-action). The stable module category $\operatorname{StMod} kG$ *is a*

rigidly compactly generated tensor triangulated category.

Group cohomology

Theorem (Golod, Venkov, Evens)

The cohomology ring $H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ is graded-commutative and noetherian.

We set

$$\mathrm{Spec}(H^*(G,k)) := \text{set of homogeneous prime ideals}$$

and

$$\mathrm{Proj}(H^*(G,k)) := \mathrm{Spec}(H^*(G,k)) \smallsetminus \{H^+(G,k)\}.$$

Local cohomology and support

For every homogeneous prime ideal $\mathfrak{p}$ of $H^*(G,k)$ there is a local cohomology functor

$$\varGamma_{\mathfrak{p}}\colon \operatorname{StMod} kG \longrightarrow \operatorname{StMod} kG$$

This functor is exact and preserves all coproducts. Moreover,

$$\varGamma_{\mathfrak{p}}(M) \cong \varGamma_{\mathfrak{p}}(k) \otimes_k M \quad \text{for all} \quad M \in \operatorname{StMod} kG.$$

Thus $\operatorname{Ker} \varGamma_{\mathfrak{p}}$ is a localising tensor ideal. The support of M is

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \operatorname{Proj}(H^*(G,k)) \mid \varGamma_{\mathfrak{p}}(M) \neq 0\}.$$

Stratification

Theorem (Benson–Iyengar–K, 2011)

The map

$$\mathsf{StMod}\,kG \supseteq \mathcal{C} \longmapsto \mathsf{supp}(\mathcal{C}) := \bigcup_{M \in \mathcal{C}} \mathsf{supp}(M)$$

induces a bijection between the

- *localising tensor ideals of* $\mathsf{StMod}\,kG$*, and*
- *subsets of* $\mathsf{Proj}(H^*(G,k))$*.*

Thus the compactly generated tensor triangulated categories

$$\begin{aligned} \varGamma_{\mathfrak{p}}(\mathsf{StMod}\,kG) &:= \{M \in \mathsf{StMod}\,kG \mid \mathsf{supp}(M) \subseteq \{\mathfrak{p}\}\} \\ &= \{M \in \mathsf{StMod}\,kG \mid M \cong \varGamma_{\mathfrak{p}}(M)\} \end{aligned}$$

are the minimal building blocks of $\mathsf{StMod}\,kG$.

Part II: Stratification – more abstractly

We think of $\mathcal{T} = \mathsf{StMod}\, kG$ as fibred over $X = \mathsf{Proj}(H^*(G,k))$.

Set-up (The triangulated category)

- $\mathcal{T}$ a compactly generated triangulated category
- $\mathcal{T}^c$ full subcategory of compact objects
- $\mathsf{Loc}(\mathcal{T})$ lattice of localising subcategories of $\mathcal{T}$
- $\mathsf{Thick}(\mathcal{T}^c)$ lattice of thick subcategories of $\mathcal{T}^c$

Set-up (The space)

- X a topological space
- $\Omega(X)$ lattice/frame of open subsets

Triangulated categories fibred over a space

Definition

$\mathcal{T}$ is fibred over X via a map $\tau\colon \Omega(X) \to \mathsf{Thick}(\mathcal{T}^c)$ if

- τ preserves finite meets and arbitrary joins, and
- $\tau(\mathsf{U})$ and $\tau(\mathsf{V})$ commute for each pair $\mathsf{U}, \mathsf{V} \in \Omega(X)$.

Example

Let R be a graded-commutative ring acting on $\mathcal{T}$. Then $\mathcal{T}$ is fibred over

$$(\operatorname{Spec} R)^{\vee} := \text{Hochster dual of } \operatorname{Spec} R.$$

Example

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be a rigidly compactly generated tensor triangulated category. Then $\mathcal{T}$ is fibred over

$$(\operatorname{Spc} \mathcal{T}^c)^{\vee} := \text{Hochster dual of the Balmer spectrum } \operatorname{Spc} \mathcal{T}^c.$$

Triangulated categories fibred over a space

Set-up

$\mathcal{T}$ is fibred over X via a map $\tau\colon \Omega(X) \to \mathsf{Thick}(\mathcal{T}^c)$.

For each $U \in \Omega(X)$ set

$$\mathcal{T}_U := \text{ localising subcategory of } \mathcal{T} \text{ generated by } \tau(U).$$

An inclusion $U \subseteq V$ induces a functor $\mathcal{T}_V \to \mathcal{T}_U$.
For each $M \in \mathcal{T}$ there is a localisation triangle

$$\Gamma_U(M) \longrightarrow M \longrightarrow L_U(M) \longrightarrow$$

with $\Gamma_U(M) \in \mathcal{T}_U$ and $L_U(M) \in (\mathcal{T}_U)^\perp$. Commutativity means

$$\Gamma_U \Gamma_V \cong \Gamma_{U\cap V} \cong \Gamma_V \Gamma_U \quad \text{for all} \quad U, V \in \Omega(X).$$

Support

Let $Y \subseteq X$ be locally closed, so $Y = V \smallsetminus U$ with $U, V \in \Omega(X)$. Set

$$\varGamma_Y := \varGamma_V L_U \qquad \text{and} \qquad \mathcal{T}_Y := \mathcal{T}_V \cap (\mathcal{T}_U)^{\perp}.$$

Lemma

The definitions of $\varGamma_Y$ and $\mathcal{T}_Y$ do not depend on U, V.

Assume: Each point $p \in X$ is locally closed. Set $\varGamma_p := \varGamma_{\{p\}}$.

Definition

For $M \in \mathcal{T}$ the support equals

$$\operatorname{supp}_{\mathcal{T}}(M) := \{p \in X \mid \varGamma_p(M) \neq 0\}.$$

This generalises definitions of Benson–Iyengar–K (2008) and Balmer–Favi (2011).

Stratification – a diagram

Set

$$P(X) := \text{power set of } X$$

and for $U \subseteq X$

$$\bar{\tau}(U) := \{M \in \mathcal{T} \mid \mathrm{supp}_\tau(M) \subseteq U\}.$$

Suppose there is an adjoint pair of maps between posets:

$$\mathrm{Thick}(\mathcal{T}^c) \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \Omega(X)$$

$$\mathcal{U} \subseteq \tau(V) \iff \sigma(\mathcal{U}) \subseteq V \qquad \text{for} \qquad \mathcal{U} \subseteq \mathcal{T}^c, V \subseteq X.$$

Then we obtain a commutative diagram of poset morphisms:

$$\begin{tikzcd}
\mathrm{Loc}(\mathcal{T}) \arrow[r, shift left, "\mathrm{supp}_\tau"] & P(X) \arrow[l, shift left, "\bar{\tau}"] \arrow[d, shift left, "\mathrm{int}"] \\
\mathrm{Thick}(\mathcal{T}^c) \arrow[u, shift left, "\mathrm{loc}"] \arrow[r, shift left, "\sigma"] & \Omega(X) \arrow[l, shift left, "\tau"] \arrow[u, shift left, "\mathrm{inc}"]
\end{tikzcd}$$

Stratification – comments

$$\begin{tikzcd}
\mathsf{Loc}(\mathcal{T}) \arrow[r, shift left, "\mathsf{supp}_\tau"] \arrow[d, shift left] & P(X) \arrow[l, shift left, "\bar{\tau}"] \arrow[d, shift left, "\mathrm{int}"] \\
\mathsf{Thick}(\mathcal{T}^c) \arrow[u, shift left, "\mathsf{loc}"] \arrow[r, shift left, "\sigma"] & \Omega(X) \arrow[l, shift left, "\tau"] \arrow[u, shift left, "\mathrm{inc}"]
\end{tikzcd}$$

- The map σ describes the support of compact objects, so $\sigma(x) = \sigma(\mathsf{thick}(x))$ for each $x \in \mathcal{T}^c$.
- Commutativity means: the square of left adjoints and the square of right adjoints both commute. In particular, $\sigma(x) = \mathsf{supp}_\tau(x)$ for each $x \in \mathcal{T}^c$.
- The map $\bar{\tau}$ is injective iff τ is injective.
- Stratification means: the map $\bar{\tau}$ identifies the subsets of X with certain localising subcategories of $\mathcal{T}$.
- There are local criteria for stratification: the local-to-global principle and a minimality condition for each $\mathcal{T}_{\{p\}}$ $(p \in X)$.

Part III: Compact and dualisable objects

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category. For $M \in \mathcal{T}$ write

$$\mathcal{H}om(M, -) := \text{ right adjoint of } M \otimes -$$

Definition

An object $M \in \mathcal{T}$ is

- compact if $\mathrm{Hom}(M, -)$ preserves all coproducts, and
- dualisable or rigid if

$$\mathcal{H}om(M, \mathbb{1}) \otimes N \xrightarrow{\sim} \mathcal{H}om(M, N) \quad \text{for all} \quad N \in \mathcal{T}.$$

$\mathcal{T}$ is rigidly compactly generated if compact $=$ dualisable.

Compact versus dualisable

Lemma

For $M \in \operatorname{StMod} kG$ are equivalent:

- *M is finite dimensional (up to an isomorphism);*
- *M is compact;*
- *M is dualisable.*

We write $\operatorname{stmod} kG$ for the full subcategory of compact objects.

Lemma

For $M \in \Gamma_{\mathfrak{p}}(\operatorname{StMod} kG)$ we have

$$M \text{ compact} \quad \Longrightarrow \quad M \text{ dualisable.}$$

The converse only holds when $\mathfrak{p}$ is a minimal prime.

Main theorem

Theorem (Benson–Iyengar–K–Pevtsova, 2024)

For $\mathfrak{p} \in \mathrm{Proj}(H^*(G,k))$ and $M \in \mathcal{T} = \varGamma_{\mathfrak{p}}(\mathrm{StMod}\, kG)$ the following are equivalent:

(1) *M has finite length in $\mathcal{T}$;*

(2) *M is dualisable in $\mathcal{T}$;*

(3) *M is in the thick subcategory generated by $\varGamma_{\mathfrak{p}}(C)$ for $C \in \mathrm{stmod}\, kG$.*

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) hold in a broader context and have been established in stable homotopy theory by Hovey and Strickland [Morava K-theories and localisation, 1999].

Finite length

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category. For $M, N \in \mathcal{T}$ write

$$\mathrm{Hom}^*(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(M, \Sigma^n N).$$

This is a graded module over $R := \mathrm{End}^*(\mathbb{1})$. So $\mathcal{T}$ is R-linear.

Definition

An object $M \in \mathcal{T}$ has finite length if $\mathrm{Hom}^*(C, M)$ has finite length over R for all compact $C \in \mathcal{T}$.

Proposition

Suppose $\mathcal{T}$ has a compact generator. Then the finite length objects form a thick subcategory that has the Krull–Schmidt property.

Proof of the main theorem

Theorem (Benson–Iyengar–K–Pevtsova, 2024)

For $\mathfrak{p} \in \mathrm{Proj}(H^*(G,k))$ and $M \in \mathcal{T} = \Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG)$ the following are equivalent:

(1) *M has finite length in $\mathcal{T}$;*

(2) *M is dualisable in $\mathcal{T}$;*

(3) *M is in the thick subcategory generated by $\Gamma_{\mathfrak{p}}(C)$ for $C \in \mathrm{stmod}\, kG$.*

The proof of the implication (1) $\Rightarrow$ (3) is based on

local regularity

for compactly generated tensor triangulated categories.

Strong generation

Let $\mathcal{C}$ be an essentially small triangulated category.

Definition (Bondal–van den Bergh, 2003)

An object $C \in \mathcal{C}$ is a

- generator if $\mathcal{C} = \mathrm{thick}(C)$ (using suspensions and extensions),
- strong generator if $\mathcal{C} = \mathrm{thick}(C)$ with a global bound on the number of extensions that are needed.

Example

Let A be a right noetherian ring. Then

$$\mathrm{Perf}(A) \text{ has a strong generator} \quad \Longleftrightarrow \quad \mathrm{gl.dim}\, A < \infty.$$

Local regularity

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be a rigidly compactly generated tensor triangulated category with $R = \operatorname{End}^*(\mathbb{1})$. Suppose that $\mathcal{T}$ is noetherian, so $\operatorname{Hom}^*(C, D)$ is a noetherian R-module for all compact $C, D \in \mathcal{T}$.

Definition

The category $\mathcal{T}$ is locally regular if $\mathcal{T}$ admits a compact generator C such that $\operatorname{thick}(\varGamma_{\mathfrak{p}}(C))$ is strongly generated for each homogeneous prime ideal $\mathfrak{p}$ of R.

Example

Let A a commutative noetherian ring. Then

$$A \text{ is regular} \quad \Longleftrightarrow \quad \mathbf{D}(\operatorname{Mod} A) \text{ is locally regular.}$$

A recollement of TT categories

Let $\mathbf{K}(\operatorname{Inj} kG)$ denote the category of complexes of injective kG-modules, up to homotopy. Consider the following recollement:

$$\begin{array}{ccccc} \operatorname{StMod} kG & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & \mathbf{K}(\operatorname{Inj} kG) & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & \mathbf{D}(\operatorname{Mod} kG) \\ \\ \operatorname{stmod} kG & \longleftarrow & \mathbf{D}^b(\operatorname{mod} kG) & \longleftarrow & \mathbf{K}^b(\operatorname{proj} kG) \\ \\ \operatorname{Proj}(H^*(G,k)) & & \operatorname{Spec}(H^*(G,k)) & & \{H^+(G,k)\} \end{array}$$

There are three levels:

- compactly generated tensor triangulated categories
- subcategories of compact objects
- points classifying the localising tensor ideals

$\mathbf{K}(\mathsf{Inj}\, kG)$ is locally regular

Theorem (Benson–Iyengar–K–Pevtsova, 2024)

The tensor triangulated category $\mathbf{K}(\mathsf{Inj}\, kG)$ is locally regular.

- The proof reduces to the case of an elem. abelian p-group, so eventually to a statement about commutative noetherian rings (via a dg Bernstein–Gelfand–Gelfand correspondence).
- For $\mathfrak{p} \in \mathrm{Proj}(H^*(G, k))$ we have

$$\varGamma_{\mathfrak{p}}(\mathbf{K}(\mathsf{Inj}\, kG)) \xrightarrow{\sim} \varGamma_{\mathfrak{p}}(\mathsf{StMod}\, kG).$$

The category of dualisable objects

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category. Set

$$\mathcal{T}^d := \text{ subcategory of dualisable objects.}$$

- The category $\mathcal{T}^d = (\mathcal{T}^d, \otimes, \mathbb{1})$ is tensor triangulated and the thick tensor ideals form a spatial frame.
- Let $\mathrm{Spc}(\mathcal{T}^d)$ denote the space of prime ideals, i.e. the Balmer spectrum of $\mathcal{T}^d$.

Theorem (Balmer, 2010)

Suppose that $\mathrm{End}^*(\mathbb{1})$ *is noetherian. Then there is a continuous and surjective comparison map*

$$\mathrm{Spc}(\mathcal{T}^d) \longrightarrow \mathrm{Spec}(\mathrm{End}^*(\mathbb{1})).$$

Local duality

Problem

Compute the Balmer spectrum of $\mathit{\Gamma}_{\mathfrak{p}}(\mathbf{K}(\operatorname{Inj} kG))^d$.

Let $R = H^*(G,k)$. For $\mathfrak{p} \in \operatorname{Spec}(R)$ set

$$\begin{aligned} R_{\mathfrak{p}} &:= \text{localisation at } \mathfrak{p} \\ R_{\mathfrak{p}}^{\wedge} &:= \text{completion of } R_{\mathfrak{p}} \text{ at its maximal ideal.} \end{aligned}$$

The tensor unit of $\mathit{\Gamma}_{\mathfrak{p}}(\mathbf{K}(\operatorname{Inj} kG))$ is $\mathit{\Gamma}_{\mathfrak{p}}(k)$.

Theorem (Benson–Iyengar–K–Pevtsova, 2019)

For $\mathfrak{p} \in \operatorname{Proj}(R)$ *we have*

$$\operatorname{End}^*(\mathit{\Gamma}_{\mathfrak{p}}(k)) \cong R_{\mathfrak{p}}^{\wedge}.$$

Balmer spectrum of $\Gamma_{\mathfrak{p}}(\mathbf{K}(\operatorname{Inj} kG))^d$

For the maximal ideal $\mathfrak{p} = H^+(G, k)$ we have

- $\Gamma_{\mathfrak{p}}(\mathbf{K}(\operatorname{Inj} kG))^d \simeq \mathbf{D}^b(\operatorname{mod} kG)$
- $H^*(G, k) \cong H^*(G, k)^{\wedge}_{\mathfrak{p}}$
- The comparison map is bijective, thanks to the classification theorem of Benson–Carlson–Rickard (1997).

Theorem (Benson–Iyengar–K–Pevtsova, 2025)

For $\mathfrak{p} \in \operatorname{Proj}(H^*(G, k))$ *the comparison map*

$$\operatorname{Spc}(\Gamma_{\mathfrak{p}}(\operatorname{StMod} kG)^d) \longrightarrow \operatorname{Spec}(H^*(G, k)^{\wedge}_{\mathfrak{p}})$$

is bijective.

Conclusion

Locally, we may think of the dualisable object as a completion of the compact objects.