Contemporary Mathematics

Weighted orbital integrals

Werner Hoffmann

ABSTRACT. We explain the definition of weighted orbital integrals and the associated invariant distributions, which appear in the Arthur-Selberg trace formula. We report their known properties, leaving aside questions of endoscopy and concentrating on the calculation of their Fourier transforms. We also announce a new result on those Fourier transforms over the reals. Finally, we give explicit Fourier transforms for the groups $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GL}(3,\mathbb{R})$.

Introduction

The present article is an introduction to weighted orbital integrals, which appear on the geometric side of the noninvariant Arthur-Selberg trace formula. We also survey the associated invariant distributions, which take their place in the invariant trace formula. We do not treat the stable versions of those distributions and related questions of endoscopy, but concentrate on the calculation of the Fourier transforms of the invariant distributions. A classical application is the determination of Gamma factors of zeta functions of Selberg type. A detailed knowledge of those Fourier transforms may also have applications in connection with Langlands' ideas [**19**] to go beyond endoscopy.

Weighted orbital integrals arise as follows. The derivation of the trace formula requires truncation of an integral kernel to make it integrable over a non-compact quotient. On the geometric side, this produces weighted orbital integrals, which are integrals of a test function on a reductive group G with respect to a certain non-invariant measure supported on a conjugacy class. On the spectral side, truncation gives rise to weighted characters, which are traces of induced representations of G twisted by certain logarithmic derivatives of intertwining operators.

In this way one gets two families, each indexed by the Levi subgroups M of G, of distributions which are non-invariant under inner automorphisms in a parallel pattern. Arthur constructed from those two families a new family of invariant distributions I_M, in terms of which the trace formula can be restated. The ordinary orbital integrals reappear unchanged as the terms I_G.

An important technical device are the differential equations satisfied by I_M as a function of the orbit in case of the groundfield of real numbers. In fact, I_M is

2000 *Mathematics Subject Classification.* Primary 22E30 22E50; Secondary 11F72.

Partly supported by the SFB 701 of the German Research Foundation.

fully characterised by those differential equations together with the asymptotic at infinity and at the singular orbits. We announce a formula for the Fourier transform of I_M restricted to the principal series when M is a split maximal torus in G.

The differential equations form a holonomic system with a regular singularity at infinity similar to the system satisfied by the matrix coefficients of an admissible representation. Thereby the Fourier transform of I_M can be explicitly calculated in terms of higher-dimensional hypergeometric series at least for groups of low real rank. We conclude the article with explicit formulas for those Fourier transforms for the groups $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GL}(3,\mathbb{R})$.

1. Definition

Let F be a local field of characteristic zero and G a connected reductive group defined over F. To simplify notation, we will use the same symbol G for the set of F-rational points of G. We will denote by A_G the greatest F-split torus in the center of G and by G^1 the intersection of the kernels of all continuous homomorphisms $G \to \mathbb{R}$.

If $F = \mathbb{R}$, then for each $g \in G$ there is a unique $H \in \mathfrak{a}_G$ such that the image of g in $G/G^1 \cong A_G/A_G^1$ equals $\exp H$. For a general local field F, we define an $\mathbb{R}$-vector space $\mathfrak{a}_G^* := \mathrm{Hom}(G,\mathbb{R})$ and denote its dual space by $\mathfrak{a}_G$. Again for each $g \in G$, there is a unique $H \in \mathfrak{a}_G$ such that every $\lambda \in \mathfrak{a}_G^*$, considered as a homomorphism $G \to \mathbb{R}$, takes the value $\lambda(H)$ on g. This gives rise to a continuous homomorphism $H_G : G \to \mathfrak{a}_G$.

Every unramified character of G, i. e., every continuous homomorphism $G \to \mathbb{C}^\times$ factoring through $\mathbb{R}_+$, is of the form $g \mapsto e^{\lambda(H_G(g))}$ for some λ in the complexified space $\mathfrak{a}_{G,\mathbb{C}}^*$. We get an action $(\lambda, \pi) \mapsto \pi_\lambda$ of the purely imaginary subspace $i\mathfrak{a}_G^*$ on the unitary dual $\Pi(G)$, i. e., the set of equivalence classes of irreducible unitary representations, by setting $\pi_\lambda(g) = e^{\lambda(H_G(g))}\pi(g)$.

Example 1.1. If $G = \mathrm{GL}(n,F)$ and we identify $A_G = F^\times$, $\mathfrak{a}_G = \mathbb{R}$, then $H_G(g) = \frac{1}{n}\log|\det g|$.

Every parabolic F-subgroup P of G has a Levi decomposition $P = MN$, where N is the unipotent radical of P and M is a connected reductive F-group just like G. We fix special maximal subgroup K, then $G = PK = MNK$. Writing $H_P(mnk) := H_M(m)$, we obtain a continuous map $H_P : G \to \mathfrak{a}_M$.

The set $\mathcal{P}(M)$ of parabolic F-subgroups P with given Levi component M is in bijection with set of chambers $\mathfrak{a}_P^+$ in $\mathfrak{a}_M$. We fix an invariant measure η_M^G on $\mathfrak{a}_M^G := \mathfrak{a}_M/\mathfrak{a}_G$ and consider, for every $x \in G$, the following volume of a convex hull:

$$v_M(x) := \mathrm{vol}_{\eta_M^G} \operatorname{conv}\{-H_P(x) : P \in \mathcal{P}(M)\}. \tag{1.1}$$

This function is left M-invariant, because $H_P(mx) = H_M(m) + H_P(x)$.

Definition 1.2. For f in the Schwartz space $\mathcal{C}(G)$ of rapidly decreasing L^2-functions on G and for $m \in M$ such that the centraliser G_m of m in G is contained in M, the weighted orbital integral is defined as

$$J_M(m,f) := |D(m)|^{1/2} \int_{G_m\backslash G} f(x^{-1}mx)v_M(x)\,d\dot{x},$$

where $D(m) := \det_{\mathfrak{g}_s\backslash\mathfrak{g}}(\mathrm{Id} - \mathrm{Ad}(s))$ if s denotes the semisimple component of m.

Here we have fixed, beside η_M^G, an invariant measure on $G_m\backslash G$, and $\mathfrak{g}$ denotes (unlike $\mathfrak{a}_G$) the Lie algebra of G over F. It has been shown ([**3**], Lemma 8.1) that the integral converges and defines a tempered distribution $J_M(m)$ on G, i.e., a continuous linear functional on $\mathcal{C}(G)$. Note that v_G is constant equal to 1, so that $J_G(g)$ is the ordinary (unweighted) orbital integral. This is the only case in which $J_M(m)$ is invariant (under inner automorphisms).

In the above discussion, one may replace F by a number field and G by the group of adèlic points. The resulting global weighted orbital integrals for F-rational points m in M and a Schwartz-Bruhat function f are the main terms on the geometric side of the trace formula ([**2**], p. 951). Global weighted orbital integrals can be reduced to their local counterparts by splitting formulas ([**6**], Prop. 9.1) and will not be discussed here.

The characteristic function of the convex hull in (1.1) can be written as an alternating sum of characteristic functions of simplicial cones indexed by the elements of $\mathcal{P}(M)$. Their Fourier transforms as functions of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ converge for $\operatorname{Re}\lambda$ in a certain chamber of $\mathfrak{a}_M$, and in the limit one obtains

$$v_M(x) = \lim_{\lambda\to 0} \sum_{P\in\mathcal{P}(M)} \frac{e^{-\lambda(H_P(x))}}{\theta_P(\lambda)}, \tag{1.2}$$

where θ_P is the suitably normalised product of the linear functions defining the walls of the dual cone ${}^+\mathfrak{a}_P^*$ of the chamber $\mathfrak{a}_P^+$.

The integral over $G_m\backslash G$ in Definition 1.2 can be written as $[G_m : G_m^0]^{-1}$ times the integral over $G_m^0\backslash G$, thus the existence condition can be weakened to $G_m^0 \subset M$. More general weighted orbital integrals for orbits with $G_m^0 \not\subset M$ also occur in the trace formula; they have to be defined by a limit process ([**5**], p. 254) that we shall describe in a special case only. Recall that the smallest Levi subgroups properly containing M are the groups M_β, for which $\mathfrak{a}_{M_\beta}$ is the kernel of β in $\mathfrak{a}_M$, where β is a root of A_M. If $G_m^0 \subset M_\beta$, then there is a real number r such that the distribution

$$J_M(ma, f) + \frac{r}{2}\log\bigl(|a^\beta - 1|\cdot|1-a^{-\beta}|\bigr)\cdot J_{M_\beta}(ma, f) \tag{1.3}$$

has a nontrivial limit as $a \in A_M$ tends to 1 while $G_{ma}^0 \subset M$, and that limit is defined to be $J_M(m,f)$. If $F = \mathbb{R}$, we may define $a^{\beta/2}$ in the obvious way for all $a \in A_M$ with $a^\beta > 0$ and simplify

$$|a^\beta - 1|\cdot|1-a^{-\beta}| = |a^{\beta/2} - a^{-\beta/2}|^2.$$

Definition 1.2 can be made more concrete when G has F-rank one. In this case there are two parabolics P and $\bar{P}$ containing a minimal Levi subgroup M, and we may parametrise $G_m\backslash G$ by $N \times K$. Substituting $n^{-1}mn = mn'$, we can view n as a function of n' and obtain

$$J_M(m,f) = -\left|D^M(m)\right|^{1/2}\delta_P(m)\int_K\int_N f(k^{-1}mn'k)\rho_P(H_{\bar{P}}(n))\,dn'\,dk,$$

where $\delta_P(m) = e^{\rho_P(H_M(m))} = \det(\mathrm{Ad}_{\mathfrak{n}}(m))^{1/2}$ is the modular character of P and ρ_P was used to normalise the measure on $\mathfrak{a}_M$. It is standard notation to indicate by a superscript M that an object is associated to the reductive group M rather than G.

In the special case when $G = \mathrm{GL}(2, F)$ and M is the subgroup of diagonal matrices, we can explicitly calculate

$$-\rho_P\left(H_{\bar{P}}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\right) = \log \|(1, x)\|$$

in terms of a K-invariant norm on F^2.

2. Invariant Fourier transform

Given a continuous representation π of G on a complex Hilbert space V and a continuous function $f : G \to \mathbb{C}$ of compact support, one can define

$$\pi(f)v = \int_G f(g)\pi(g)v\,dg \tag{2.1}$$

for every $v \in V$. Then

$$\pi(f_1 * f_2) = \pi(f_1)\pi(f_2). \tag{2.2}$$

If π is admissible and f is smooth, which means "locally constant" in the p-adic case, the operator $\pi(f)$ is of trace class, and the map $f \mapsto \operatorname{tr}\pi(f)$ is a distribution, called the character of π. The set $G_{\mathrm{reg}} := \{g \in G : G_g^0 \text{ is a torus}\}$ of regular semisimple elements has complement of Haar measure zero. By Harish-Chandra's regularity theorem, there exists a locally integrable smooth function Θ_π on G_{reg} such that

$$\operatorname{tr}\pi(f) = \int_G \Theta_\pi(g)f(g)\,dg. \tag{2.3}$$

Note that $\Theta_{\check{\pi}} = \bar{\Theta}_\pi$, where $\check{\pi}$ denotes the contragredient representation of π. An admissible representation π is called tempered if its matrix coefficients define tempered distributions. Such a representation is unitarisable, (2.1) converges for $f \in \mathcal{C}(G)$, and the character of π is a tempered distribution as well. Sometimes it is preferable to work with the function $\Phi(\pi, g) = |D(g)|^{1/2}\Theta(g)$.

We define the Fourier transform of $f \in \mathcal{C}(G)$ as the function $\hat{f}$ on the set $\Pi_{\mathrm{temp}}(G)$ of equivalence classes of tempered representations of G by

$$\hat{f}(\pi) := \operatorname{tr}\pi(f).$$

Equation (2.2) implies

$$\widehat{f_1 * f_2} = \widehat{f_2 * f_1}. \tag{2.4}$$

The *trace Paley-Wiener theorem* [**11**] claims that $f \mapsto \hat{f}$ is an open, continuous surjection of $\mathcal{C}(G)$ onto a "Schwarz space" $\mathcal{I}(G)$ of functions $\phi : \Pi_{\mathrm{temp}}(G) \to \mathbb{C}$. This statement has substance once an independent description of the space $\mathcal{I}(G)$ is given. We will say more about that in the next section. The name of the theorem originates from its analogue [**13**] for the space $\mathcal{H}(G)$ of compactly supported smooth functions on G, also called the Hecke algebra.

Definition 2.1. The tempered distribution $I : \mathcal{C}(G) \to \mathbb{C}$ has the Fourier transform $\hat{I} : \mathcal{I}(G) \to \mathbb{C}$ if $\hat{I}(\hat{f}) = I(f)$ for all $f \in \mathcal{C}(G)$.

It follows from (2.4) that a tempered distribution I possessing a Fourier transform has the property $I(f_1 * f_2) = I(f_2 * f_1)$. By letting f_2 converge to a delta distribution, one easily shows that the latter property is equivalent to I being invariant under inner automorphisms (whence the heading). Conjecturally, every invariant tempered distribution has a Fourier transform.

Example 2.2. If I is the delta distribution at the unit element, i. e., $I(f) = J_G(e,f) = f(e)$, then $\hat{I}$ is the Plancherel measure on $\Pi_{\mathrm{temp}}(G)$. Since $\hat{f}$ is proportional to the Haar measure chosen, the Plancherel measure is inversely proportional to the Haar measure.

We mention in passing that there is also a Paley-Wiener theorem for the (bijective) operator-valued Fourier transform $\pi \mapsto \pi(f)$ and a corresponding notion of non-invariant Fourier transform of distributions.

3. Tempered dual

A unitary representation of G is called square-integrable (modulo the center) if the square of the absolute value of one (equivalently: all) of its matrix coefficients is integrable over G/A_G. We denote the set of equivalence classes of square-integrable representations by $\Pi_2(G) \subset \Pi_{\mathrm{temp}}(G)$. In the case of compact centre, i. e., trivial A_G, this is the set of all atoms of the Plancherel measure, also called the discrete series of G.

If L is a Levi component of a parabolic subgroup P of G (for short, a Levi subgroup of G) and σ a tempered representation of L, considered as a representation of P, then the induced representation $\mathrm{Ind}_P^G \sigma$ is tempered and completely reducible of finite length. Every tempered representation is a constituent of such a representation parabolically induced from some square-integrable representation σ of some Levi subgroup L.

If $f \in \mathcal{C}(G)$, then $\mathrm{tr}\,\mathrm{Ind}_P^G \sigma_\lambda(f)$ is a Schwartz function of $\lambda \in i\mathfrak{a}_L^*/\operatorname{Stab}\sigma$. Thus, for $\phi \in \mathcal{I}(G)$, $\phi(\mathrm{Ind}_P^G \sigma_\lambda)$ has to extend across the points of reducibility to a Schwartz function. Here, the stabiliser of σ is trivial for archimedean F, while it is a lattice in $i\mathfrak{a}_L^*$ for p-adic F, in which case any smooth function on the quotient torus is called a Schwartz function. Incidentally, if f belongs to the Hecke algebra $\mathcal{H}(G)$, then $\mathrm{tr}\,\mathrm{Ind}_P^G \sigma_\lambda(f)$ is a Paley-Wiener function of $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*/\operatorname{Stab}\sigma$ in the archimedean case and a finite Fourier series in the p-adic case.

If $P' = LN'$ is another parabolic with the same Levi component L, there is an operator $J_{P'|P}(\sigma_\lambda)$ intertwining $\mathrm{Ind}_P^G \sigma_\lambda$ with $\mathrm{Ind}_{P'}^G \sigma_\lambda$. It maps ψ to the function ψ' defined by

$$\psi'(g) = \int_{N \cap N' \backslash N'} \psi(n'g)\, dn',$$

where the integral converges for $\operatorname{Re}\lambda$ in some cone in $\mathfrak{a}_L^*$. By restriction $\psi|_K$, the space of $\mathrm{Ind}_P^G \sigma_\lambda$ is realised independently of λ as a space of functions on K with values in the space of σ, called the compact picture. For K-finite ψ, the vector $J_{P'|P}(\sigma_\lambda)\psi$ has a meromorphic continuation to $\mathfrak{a}_{L,\mathbb{C}}^*$.

One can choose nonzero meromorphic functions $r_{P'|P}$ as in ([**8**], Theorem 2.1) so that

$$J_{P'|P}(\sigma_\lambda) = r_{P'|P}(\sigma_\lambda) R_{P'|P}(\sigma_\lambda), \tag{3.1}$$

where the normalised intertwining operator $R_{P'|P}(\sigma_\lambda)$ is holomorphic at $\lambda = 0$ and satisfies

$$R_{P''|P'}(\sigma) R_{P'|P}(\sigma) = R_{P''|P}(\sigma), \qquad R_{P'|P}(\sigma)^* = R_{P|P'}(\sigma). \tag{3.2}$$

In particular, $R_{P'|P}(\sigma)$ is unitary, hence invertible. Thus, for the generic $\lambda \in i\mathfrak{a}_L^*$ for which the representation

$$\sigma^G := \mathrm{Ind}_P^G \sigma$$

is irreducible, it does not depend on $P \in \mathcal{P}(L)$.

The trace of a representation induced from a subgroup is supported on the conjugacy classes that meet the subgroup. It follows that Θ_{σ^G} for a representation σ of a proper Levi subgroup vanishes on the set G_{ell} of all elliptic semisimple elements, i. e., elements that are not contained in any proper parabolic subgroup.

Example 3.1. Let $I(f) = J_G(l, f)$, where $l \in L \cap G_{\mathrm{reg}}$. This distribution is proportional to the measure chosen on $G_l \backslash G$, hence the Fourier transform $\hat{I}(\phi) = \hat{J}_G(l, \phi)$ must be inversely proportional to the measure on G_l. If $\phi \in \mathcal{I}(G)$ is such that $\operatorname{supp} \phi \subset \{\sigma^G : \sigma \in \Pi_2(L)\}$, then $\hat{J}_G(l, \phi)$ vanishes unless $l \in L_{\mathrm{ell}}$, in which case

$$\hat{J}_G(l, \phi) = \sum_{\sigma \in \Pi_2(L)/i\mathfrak{a}_L^*} \int_{i\mathfrak{a}_L^* / \operatorname{Stab}\sigma} \Phi^L(\sigma_\lambda, l)\phi(\sigma_\lambda^G)\, d\lambda. \tag{3.3}$$

Note that for $l \in L_{\mathrm{ell}} \cap G_{\mathrm{reg}}$, the torus G_l/A_L is compact and has a Haar measure of total mass 1. Thus the measure on G_l used in the definition of $J_G(l, f)$ corresponds to a measure on A_L, and the measure in (3.3) is the corresponding Plancherel measure. We can combine sum and integral into one integral over $\Pi_2(L)$ with respect to a measure $d\sigma$ (different from the Plancherel measure of L).

We would like to rewrite the integral in the example as an integral over (a subset of) $\Pi_{\mathrm{temp}}(G)$ in line with definition 2.1. An element $w \in W_L := \operatorname{Norm}_G L/\operatorname{Cent}_G L$, whose representative $\tilde{w}$ can be chosen in K, induces a unitary intertwining operator

$$\operatorname{Ind}_{P^w}^G \sigma \sim \operatorname{Ind}_P^G (w\sigma)$$

simply by left translation, where $P^w = \tilde{w}^{-1} P \tilde{w}$ and $w\sigma(l) = \sigma(\tilde{w}^{-1} l \tilde{w})$. In fact, two representations σ^G and σ'^G are equivalent iff the pairs (L, σ) and (L', σ') are G-conjugate. For $\phi \in \mathcal{I}(G)$, we see that $\phi(\sigma^G)$ is W_L-invariant as a function of λ, and we can rewrite the right-hand side of (3.3) as an integral over a subset of $W_L \backslash \Pi_2(L)$ of full measure, which is embedded into $\Pi_{\mathrm{temp}}(G)$.

If $\phi \in \mathcal{I}(G)$, then the limit of $\phi(\sigma_\lambda^G)$ as λ tends to a point of reducibility has to coincide with the sum of $\phi(\pi)$ over the constituents π. In the $\mathbb{C}$-vector space with basis $\Pi_{\mathrm{temp}}(G)$, Arthur has defined ([**4**], p. 93) a new basis $T(G)$ that includes all the representations $\operatorname{Ind}_P^G \sigma$, $\sigma \in \Pi_2(L)$, in order to avoid the aforementioned compatibility conditions in the description of $\mathcal{I}(G)$. The subset

$$T_{\mathrm{ell}}(G) := \{\tau \in T(G) : \Theta_\tau|_{G_{\mathrm{ell}}} \neq 0\}$$

is a union of $i\mathfrak{a}_G^*$-orbits, and the map $\tau \mapsto \tau^G$ embeds $W_L \backslash T_{\mathrm{ell}}(L)$ into $T(G)$, the latter being the disjoint union of the images over all conjugacy classes of Levi subgroups L. In this way, one gets a description

$$\mathcal{I}(G) \cong \widehat{\bigoplus_{[L]}} \mathcal{C}(T_{\mathrm{ell}}(L))^{W_L}$$

as a direct sum of Weyl group invariants in Schwartz spaces completed with respect to a natural topology.

Example 3.2. For $\mathrm{SL}(2, \mathbb{R})$, in order to pass from $\Pi_{\mathrm{temp}}(G)$ to $T(G)$, one has to replace the limits of discrete series π_1, π_2 by the virtual representations $\tau_\pm = \pi_1 \pm \pi_2$. Here τ_+ is induced, while $\tau_- \in T_{\mathrm{ell}}(G)$.

4. Weighted characters

While weighted orbital integrals appear on the geometric side of the trace formula, weighted characters are the analogous terms on the spectral side. For simplicity, we discuss them for tempered representations of reductive groups over local fields only. Just like the exponential functions in (1.2), intertwining operators provide another instance of Arthur's notion of (G, M)-families (in the given case, (G, L)-families, because we reserve the letter M for the geometric side and the letter L for the spectral side).

Definition 4.1. For a Levi subgroup L of G and $\sigma \in \Pi_{\mathrm{temp}}(L)$, set

$$\mathcal{R}_P(\sigma) := \lim_{\lambda \to 0} \sum_{P' \in \mathcal{P}(L)} \frac{R_{P'|P}(\sigma)^{-1} R_{P'|P}(\sigma_\lambda)}{\theta_P(\lambda)},$$

where the intertwining operators are considered in the compact picture (on our fixed maximal compact subgroup K) of $\operatorname{Ind}_P^G \sigma_\lambda$, which does not depend on λ.

If L is maximal, we can write this operator as a logarithmic derivative:

$$\mathcal{R}_P(\sigma) = -\frac{1}{\theta_P(\lambda)} R_{\bar{P}|P}(\sigma)^{-1} \frac{d}{dz} R_{\bar{P}|P}(\sigma_{z\lambda})\Big|_{z=0}.$$

Although $\mathcal{R}_P(\sigma)$ is a priori only defined on K-finite vectors, and the weighted character

$$\phi_L(f, \sigma) := \operatorname{tr}(\operatorname{Ind}_P^G(\sigma, f) \mathcal{R}_P(\sigma))$$

only for $f \in \mathcal{H}(G)$, one can show ([**10**], p. 175) that it extends to $f \in \mathcal{C}(G)$ and provides a continuous map

$$\phi_L : \mathcal{C}(G) \to \mathcal{I}(L).$$

It follows from (3.2) and the intertwining property of $R_{P'|P}$ that ϕ_L is independent of $P \in \mathcal{P}(L)$, as the notation suggests. By definition, we have

$$\phi_G(f) = \hat{f}. \tag{4.1}$$

The normalising factors $r_{P'|P}(\sigma_\lambda)$ give rise to a scalar-valued (G, L)-family if they have no poles at $\lambda = 0$. However, that happens because they have to cancel the poles of the intertwining operators $J_{P'|P}(\sigma_\lambda)$ for $\lambda \in i\mathfrak{a}_L^*$.

Definition 4.2. For a Levi subgroup L of G and generic $\sigma \in \Pi_{\mathrm{temp}}(L)$, set

$$r_P(\sigma) := \lim_{\lambda \to 0} \sum_{P' \in \mathcal{P}(L)} \frac{r_{P'|P}(\sigma)^{-1} r_{P'|P}(\sigma_\lambda)}{\theta_P(\lambda)}.$$

Then $r_P(\sigma_\lambda)$ extends to a meromorphic function of $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$. It will turn up in our formulas for Fourier transforms.

5. Invariant distributions

Neither weighted orbital integrals nor weighted characters are invariant when their subscripts are proper Levi subgroups, hence they have no Fourier transform in the sense of Definition 2.1. Now since they appear on the two sides of the trace formula, it does not come as a surprise that they behave similarly under inner automorphisms. Interpreting the members of one of the families as transformations, one can use them to modify the other family to produce invariant distributions I_M. These are the main terms in the invariant trace formula [**7**]. There are two choices which are reciprocal in a certain sense ([**3**], p. 7/8, [**10**], §3). It is analytically easier to modify the weighted orbital integrals.

The ordinary orbital integral is already invariant, so one simply sets

$$I_G(g, f) := J_G(g, f).$$

For Levi subgroups M that are maximal in G, one can easily single out a non-invariant part of the weighted orbital integral $J_M(m, f)$ by composing the map ϕ_M with the Fourier transform of the invariant orbital integral on M: If we write

$$J_M(m, f) = \hat{J}_M^M(m, \phi_M(f)) + I_M(m, f),$$

then the remaining term, denoted $I_M(m, f)$, turns out to be invariant. If, moreover, M is minimal, i. e., M/A_M is compact, the Fourier transform can be made explicit using Example 3.1:

$$J_M(m, f) = \int_{\Pi_{\mathrm{temp}}(M)} \Phi^M(\sigma, m)\phi_M(f, \sigma)\, d\sigma + I_M(m, f). \tag{5.1}$$

Arthur has found a generalisation to arbitrary Levi subgroups.

Theorem 5.1. ([**3**], p. 53, [**10**], p. 179) *For all G, M as in Definition 1.2 and $m \in M \cap G_{\mathrm{reg}}$, there are invariant tempered distributions $I_M(m) = I_M^G(m)$ on G such that*

$$J_M(m, f) = \sum_{M' \supset M} \hat{I}_M^{M'}(m, \phi_{M'}(f))$$

for all $f \in \mathcal{C}(G)$, where the sum is taken over all Levi subgroups M' of G containing M.

The term with $M' = G$ in the sum simplifies in view of (4.1), and the equivalent form

$$J_M(m, f) = I_M(m, f) + \sum_{\substack{M' \supset M \\ M' \neq G}} \hat{I}_M^{M'}(m, \phi_{M'}(f))$$

of the equation can be read as a recursive definition of $I_M(m, f)$ in terms of the invariant distributions on groups M' of smaller semisimple rank than G. The existence of the Fourier transforms is part of the theorem. This is the hardest part of the proof and relies on the local or global trace formula. The invariance of $I_M(m, f)$ however is straightforward, as is its independence of the choice of a maximal compact subgroup K.

In the case $A_G = \{e\}$, the set $T_{\mathrm{ell}}(G)$ is the discrete part of $T(G)$, and a linear functional on $\mathcal{I}(G)$, evaluated on functions with support in $T_{\mathrm{ell}}(G)$, yields a linear combination of delta-distributions on $T_{\mathrm{ell}}(G)$. In the Fourier transform of $I_M(m)$, however, delta-distributions at several other points may occur, giving rise to a larger set $T_{\mathrm{disc}}(G) \supset T_{\mathrm{ell}}(G)$, which can be explicitly described. In general, $T_{\mathrm{disc}}(G)$ is $i\mathfrak{a}_G^*$-invariant, and a measure on this set can be defined in analogy with Example 3.1.

Theorem 5.2. ([**10**], p 183) *There exist smooth functions $\Phi_{M,L}(m, \tau)$ of $m \in M \cap G_{\mathrm{reg}}$ and $\tau \in T_{\mathrm{disc}}(L)$ such that, for all $\phi \in \mathcal{I}(G)$,*

$$\hat{I}_M(m, \phi) = \sum_{[L]} \int_{W_L \backslash T_{\mathrm{disc}}(L)} \Phi_{M,L}(m, \check{\tau})\phi(\tau^G)\, d\tau.$$

In Arthur's statement, the sum runs over all Levi subgroups containing a fixed minimal one, and the integral is over the full set $T_{\mathrm{disc}}(L)$. This explains the additional factor $|W_0^L|/|W_0^G|$ in his formula.

The distributions $I_M(m)$ satisfy descent identities ([**10**], (3.7)). If M_1 is a Levi subgroup of G containing M, then

$$I_{M_1}(m,f) = \sum_{M_2 \supset M} d_M^G(M_1,M_2)\hat{I}_M^{M_2}(m,\hat{f}_{M_2}),$$

where the sum is over all Levi subgroups M_2 of G containing M, and $\hat{f}_{M_2}(\tau_2) = \hat{f}(\tau_2^G)$ for $\tau_2 \in T(M_2)$. The constant $d_M^G(M_1,M_2)$ vanishes unless the natural map $\mathfrak{a}_M^{M_2} \to \mathfrak{a}_M^G/\mathfrak{a}_M^{M_1} \cong \mathfrak{a}_{M_1}^G$ is an isomorphism, in which case $\eta_{M_1}^G$ equals $d_M^G(M_1,M_2)$ times the image of $\eta_M^{M_2}$. (If η_M^G is compatible with $\eta_M^{M_1}$ and $\eta_{M_1}^G$, this can be equivalently expressed as follows: the constant vanishes unless the sum map $\mathfrak{a}_M^{M_1} \oplus \mathfrak{a}_M^{M_2} \to \mathfrak{a}_M^G$ is an isomorphism, in which case η_M^G equals $d_M^G(M_1,M_2)$ times the image of the product of $\eta_M^{M_1}$ and $\eta_M^{M_2}$.) With Theorem 5.2 it follows (cf. [**10**], (4.3)) that

$$\Phi_{M_1,L}(m,\tau) = \sum_{M_2 \supset M} d_M^G(M_1,M_2) \sum_{[L_2]_{M_2}} \sum_{w \in W_{L_2}^{M_2} \backslash W_{L_2,L}} \Phi_{M,L_2}^{M_2}(m,w\tau), \tag{5.2}$$

where the middle sum is over M_2-conjugacy classes of Levi subgroups L_2 of M_2, and $W_{L_2,L} = \{g \in G \mid gLg^{-1} = L_2\}/L = L_2\backslash\{g \in G \mid gLg^{-1} = L_2\}$.

Theorem 5.2 was proved in [**10**] with the aid of the local trace formula. This approach yields also a reciprocity between weighted orbital integrals and weighted characters, whose simplest example is the following.

Theorem 5.3. ([**9**], p 106) *If $\pi \in \Pi_2(G) \subset T_{\mathrm{ell}}(G)$, then*

$$\Phi_{M,G}(m,\pi) = \begin{cases} (-1)^{\dim \mathfrak{a}_M^G}\Phi(\pi,m) & \text{for } m \in M_{\mathrm{ell}}, \\ 0 & \text{otherwise,} \end{cases}$$

In the reference, $\Phi_{M,G}(m,\tau)$ is actually computed for any $\tau \in T_{\mathrm{ell}}(G)$, while the special case stated above had been proved already in [**1**] using differential equations.

Let us restate Example 3.1 in the present notation. If $l \in L \cap G_{\mathrm{reg}}$ and $\sigma \in \Pi_2(L)$ with σ^G irreducible, then

$$\Phi_{G,L}(l,\sigma) = \begin{cases} \sum_{w \in W_L} \Phi^L(w\sigma,l) & \text{for } l \in L_{\mathrm{ell}}, \\ 0 & \text{otherwise.} \end{cases}$$

The full Fourier transform of $J_G(g)$, $g \in G_{\mathrm{reg}}$, for $F = \mathbb{R}$ has been determined by Herb [**15**] using character identities due to Shelstad.

The descent identities reduce the distributions $I_M(m)$ to the case when $m \in M_{\mathrm{ell}}$. This allows one to calculate $\Phi_{M,L}$ from the above results for $F = \mathbb{R}$ and a range of pairs (M,L) including those with $\dim \mathfrak{a}_M^G + \dim \mathfrak{a}_L^G \leq \dim \mathfrak{a}_{G_m}^G$. The latter have also been obtained using differential equations [**16**].

The explicit formulas mentioned so far describe the restriction of the Fourier transform $\hat{I}_M(m)$ to a certain subset of $T(G)$. If $\hat{f}$ is supported on that subset, then $I_M(m,f) = J_M(m,f)$. The remaining components of $\hat{I}_M(m)$, however, do depend on the choice of normalizing factors in (3.1).

6. Differential equations

Let the ground field F be $\mathbb{R}$. Now every $\pi \in \Pi(G)$ gives rise to a representation of the Lie algebra $\mathfrak{g}$ as well as its universal enveloping algebra $U(\mathfrak{g})$ on the subspace of smooth vectors. By a version of Schur's lemma, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by scalars, and the resulting algebra homomorphism $\chi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ is called the infinitesimal character of π.

If L is a Levi subgroup and $\sigma \in \Pi(L)$, then the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to Z(\mathfrak{l})$, $z \mapsto z_L$, is characterised by the formula $\chi_{\sigma^G}(z) = \chi_\sigma(z_L)$ for the infinitesimal character of a parabolically induced representation. This homomorphism depends only on the complexifications of $\mathfrak{g}$ and $\mathfrak{l}$ and can also be defined for a maximal torus T in place of L. In this case, we get the Harish-Chandra isomorphism from $Z(\mathfrak{g}_\mathbb{C})$ to the algebra $U(\mathfrak{t}_\mathbb{C})^W$ of invariants in $Z(\mathfrak{t}_\mathbb{C}) = U(\mathfrak{t}_\mathbb{C})$ under the action of the Weyl group W of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Each $\lambda \in \mathfrak{t}_\mathbb{C}^*$ determines a character χ_λ of $Z(\mathfrak{g}_\mathbb{C})$ by $\chi_\lambda(z) = z_T(\lambda)$, where we view $z_T \in U(\mathfrak{t}_\mathbb{C})$ as a polynomial on $\mathfrak{t}_\mathbb{C}^*$, and every character of $Z(\mathfrak{g}_\mathbb{C})$ is of this form.

The global characters Θ_π in (2.3) are invariant eigendistributions of the algebra $Z(\mathfrak{g})$, whose elements can be viewed as biinvariant differential operators on G. Harish-Chandra determined all tempered invariant eigendistributions explicitly, and as a byproduct he showed that orbital integrals satisfy certain differential equations. Arthur generalised these differential equations to the case of weighted orbital integrals and the associated invariant distributions $I_M(m)$, $m \in M \cap G_{\mathrm{reg}}$.

Since those distributions depend on m up to conjugacy only, we may assume that $m = t$ lies in a fixed maximal torus T and normalise the Haar measure on $G_t = T$ independently of $t \in T \cap G_{\mathrm{reg}}$.

THEOREM 6.1. ([**5**], p. 279) *For every connected reductive $\mathbb{R}$-group G, its Levi subgroup M and a maximal torus $T \subset M$, there exists a smooth map $\partial_M = \partial_M^G : T \cap G_{\mathrm{reg}} \to \mathrm{Hom}_\mathbb{C}(Z(\mathfrak{g}), U(\mathfrak{t}))$ with the following property. If L is a Levi subgroup of G, $\tau \in T_{\mathrm{ell}}(L)$ and $\chi = \chi_{\tau^G}$, then the functions $\Phi_M(t) = \Phi_{M,L}(t,\tau)$ satisfy the differential equations*

$$\chi(z)\Phi_M(t) = \sum_{M' \supset M} \partial_M^{M'}(t, z_{M'})\Phi_{M'}(t) \tag{6.1}$$

on $T \cap G_{\mathrm{reg}}$ for every $z \in Z(\mathfrak{g})$. Moreover, $\deg \partial_M(t,z) \le \deg z$ *and*

$$\partial_G(t,z) = z_T. \tag{6.2}$$

The smoothness of the map $\partial_M(t,z)$ is meant for fixed z, where it takes values in a finite-dimensional space. If we combine the $\partial_M^{M'}$ into a matrix, we get an algebra homomorphism from $Z(\mathfrak{g})$ into the algebra of matrix-valued differential operators on $T \cap G_{\mathrm{reg}}$. Thus, it suffices to consider (6.1) for z in a (finite) set of generators of $Z(\mathfrak{g})$.

There is an algorithm to compute the differential operators, but no general closed formula except for the Casimir element $\omega \in Z(\mathfrak{g})$ (shifted by a constant), which is characterised by $\omega_T(\lambda) = \langle \lambda, \lambda \rangle$ for a nondegenerate symmetric bilinear form on $\mathfrak{t}^*$ whose extension to $\mathfrak{t}_\mathbb{C}^*$ is W-invariant. In fact, $\partial_M(t,\omega)$ vanishes unless either $M = G$, in which case it is given by (6.2), or M is maximal, in which case

$$\partial_M(t,\omega) = \sum_\alpha \frac{|\langle \eta_M^G, \alpha \rangle|}{(t^\alpha - 1)(1 - t^\alpha)}, \tag{6.3}$$

where the sum is taken over all roots α of $\mathfrak{t}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C},\alpha} \not\subset \mathfrak{m}_{\mathbb{C}}$, and the measure η_M^G on the one-dimensional space $\mathfrak{a}_M^G$ is interpreted as a linear functional determined up to sign and extended to a linear functional on $\mathfrak{t}$ vanishing on $\mathfrak{t} \cap \mathfrak{m}^1 + \mathfrak{a}_G$.

For a parabolic subgroup $P = M_P N$ containing T, set

$$T_P := \{t \in T : |t^\alpha| > 1 \,\forall \text{ roots } \alpha \text{ of } \mathfrak{t}_{\mathbb{C}} \text{ in } \mathfrak{n}_{\mathbb{C}}\}$$

and define the subset $T_{\mathbb{C},P}$ of $T_{\mathbb{C}}$ by the same conditions. The limit of a function on $T_{\mathbb{C},P}$ as $t \underset{P}{\to} \infty$ is meant with respect to the filter of all the translates of the sets $T_{\mathbb{C},P}$.

THEOREM 6.2. ([**18**], pp. 780, 785, 790) *The system applied to $(\Phi_M)_{M \supset M_P}$ is holonomic on $T_{\mathbb{C}} \cap G_{\mathbb{C},\mathrm{reg}}$ and has a regular singularity at ∞ on $T_{\mathbb{C},P}$. For every $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ with $\chi_\lambda = \chi$ there is a unique solution $\Psi = \Psi^{P,\lambda}$ on the universal covering $\tilde{T}_{\mathbb{C},P}$ such that*

- $\Psi_G(\exp H) = e^{\lambda(H)}$,
- $\Psi_M(t) \to 0$ *as* $t \underset{P}{\to} \infty$ *if* $M \neq G$.

For sufficiently regular χ, every solution is of the form

$$\Phi_M(t) = \sum_{\chi_\lambda = \chi} \sum_{M' \supset M} c_{M'}(\lambda) \Psi_M^{P \cap M', \lambda}(t)$$

for suitable functions $c_{M'}$.

Our holonomic system is similar to the system satisfied by matrix coefficients of admissible irreducible representations studied in [**14**], but it is more complicated as the torus T need not be $\mathbb{R}$-split. By general results on holonomic systems with regular singularity, the standard solutions $\Psi_M(t)$ have a series expansion in powers of t^α, where α runs through the roots of $T_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}_{\mathbb{C}}$.

In order to compute the Fourier transforms of the invariant distributions $I_M(t)$ explicitly, it remains to solve the following problems:

(1) Find $c_{M'}(\lambda) = c^P_{M',L}(\lambda, \tau)$ for the Fourier transforms $\Phi_M(t) = \Phi_{M,L}(t, \tau)$ on each sector T_P.
(2) Describe the standard solutions Ψ_M explicitly.

7. Asymptotic formula and jump relations

We retain the notation of the preceding section and fix a parabolic $P \in \mathcal{P}(M)$. Arthur's asymptotic formula gives information about $I_M(m_T)$ for $m_T = m \exp T$ as the point T tends to infinity in the chamber $\mathfrak{a}_P^+$ in the sense that its distance from the walls grows linearly with its norm. However, $I_M(m_T, f)$ will tend to zero for any fixed $f \in \mathcal{C}(G)$, which has thus to be replaced by a function $f_T \in \mathcal{C}(G)$ that varies with T. It is characterised with the help of multipliers as follows.

If L is a Levi subgroup, $\sigma \in \Pi_2(L)$ and $Q \in \mathcal{P}(L)$, then $\mathrm{Ind}_Q^G \sigma(f_T) = 0$ unless a conjugate of L is contained in M, and if $L \subset M$, then

$$\mathrm{Ind}_Q^G \sigma(f_T) = \frac{1}{|W_L|} \sum_{w \in W_L} e^{w\nu(T)} \,\mathrm{Ind}_Q^G \sigma(f).$$

Here $\nu \in i\mathfrak{a}_L^*$ denotes the infinitesimal central character of σ defined by $\sigma(\exp H) = e^{\nu(H)}\mathrm{Id}$ for $H \in \mathfrak{a}_L$. Note that for $M = G$ one simply gets $f_T(x) = f(x(\exp T)^{-1})$.

Theorem 7.1. *If $P \in \mathcal{P}(M)$, $m \in M \cap G_{\mathrm{reg}}$ and $f \in \mathcal{H}(G)$, then*

$$\lim_{\substack{T \to \infty \\ P}} I_M(m_T, f_T) = e^{-\varepsilon(H_M(m))} \hat{I}_M^M(m, r_{P,\varepsilon} \hat{f}_{M,\varepsilon}),$$

where $\varepsilon \in (\mathfrak{a}_P^*)^+$ is sufficiently small and

$$\hat{f}_{M,\varepsilon}(\sigma) = \hat{f}(\sigma_\varepsilon^G), \qquad r_{P,\varepsilon}(\sigma) = r_P(\sigma_\varepsilon)$$

for $\sigma \in \Pi_{\mathrm{temp}}(M)$ in the notation of Definition 4.2.

If we combine this result with the Riemann-Lebesgue lemma, we can determine the coefficients $c_{M',L}^P$ under favourable circumstances. With the aid of descent formulas the result can be given a particularly simple shape in the following case.

Theorem 7.2. *If T is an $\mathbb{R}$-split maximal torus contained in a minimal parabolic P and $\sigma(t) = t^\lambda$ is sufficiently regular, then for $t \in T_P$ we have*

$$\Phi_{T,T}(t, \sigma) = \sum_{w \in W} \sum_{M \supset T} r_{P \cap M}(w\sigma) \Psi_T^{P, w\lambda}(t).$$

The details will appear in a forthcoming paper.

In general, the limit formula does not determine the solution of the differential equation uniquely. However, additional information is provided by the jump relations ([**1**], Theorem 6.1) part of which we are going to state now.

Let α be a real root of the maximal torus T and $\beta = \alpha|_{A_M}$. As a special case of (1.3), we consider the distribution

$$J_M^\alpha(t, a, f) = J_M(ta, f) + |\eta_M^{M_\beta}(\check{\alpha})| \log|a^{\beta/2} - a^{-\beta/2}| \cdot J_{M_\beta}(ta, f),$$

which is defined for $t \in T$ such that $\pm\alpha$ is the only root of T with $t^\alpha = 1$ and $a \in A_M$ sufficiently close to 1. Here we interpret $\eta_M^{M_\beta}$ as in (6.3).

Choose root vectors $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that $[X_\alpha, X_{-\alpha}] = \check{\alpha}$ and define the element $c = \exp \frac{\pi}{4i}(X_\alpha + X_{-\alpha})$ (e. g., in the complex adjoint group). The so-called Cayley transform $\mathrm{Ad}(c)$ fixes the kernel of α in T pointwise, maps $\mathfrak{t}_\mathbb{C}$ to the complexified Lie algebra of a maximal torus T_1 of M_β and the coroot $\check{\alpha} \in \mathfrak{t}$ to a coroot $\check{\alpha}_1 = i(X_\alpha - X_{-\alpha}) \in i\mathfrak{t}_1$. If we set, for $\theta \in \mathbb{R}$,

$$a_\theta = \exp(\theta\check{\alpha}) \in A_M, \qquad b_\theta = \exp(-i\theta\check{\alpha}_1) \in T_1,$$

and define the Haar measures on T and T_1 by volume forms whose complexifications correspond under c, then

$$\lim_{\theta \to 0+} \frac{d}{d\theta} J_M^\alpha(t, a_\theta, f) - \lim_{\theta \to 0-} \frac{d}{d\theta} J_M^\alpha(t, a_\theta, f) = 2|\eta_M^{M_\beta}(\check{\alpha})| \lim_{\theta \to 0+} \frac{d}{d\theta} J_{M_\beta}(tb_\theta, f).$$

We may replace J by I everywhere in equation (1.3) to define the invariant distributions $I_M^\beta(m, f)$, and the jump relations are true for them as well. Consequently, if we define

$$\Phi_{M,L}^\alpha(t, a, \tau) = \Phi_{M,L}(ta, \tau) + |\eta_M^{M_\beta}(\check{\alpha})| \log|a^{\beta/2} - a^{-\beta/2}| \cdot \Phi_{M_\beta,L}(ta, \tau),$$

then

$$\begin{aligned} (7.1) \qquad \lim_{\theta \to 0+} \frac{d}{d\theta} \Phi_{M,L}^\beta(t, a_\theta, \tau) &- \lim_{\theta \to 0-} \frac{d}{d\theta} \Phi_{M,L}^\beta(t, a_{-\theta}, \tau) \\ &= 2|\eta_M^{M_\beta}(\check{\alpha})| \lim_{\theta \to 0+} \frac{d}{d\theta} \Phi_{M_\beta,L}(tb_\theta, \tau) \end{aligned}$$

for t as above and $\tau \in T_{\rm disc}(L)$.

To determine the coefficients $c^P_{M',L}(\lambda,\tau)$, we should find the analogous jump relations for the standard solutions Ψ_M. This has been done for groups G of real rank one in [**18**] and partly for $G = \mathrm{GL}(3,\mathbb{R})$ (see section 10).

8. Explicit solutions of the holonomic system

Now we turn to Problem 2, i. e., the explicit determination of the standard solutions Ψ_M of our holonomic system. For sufficiently regular χ, the coefficients of the series Ψ_M can be determined using the differential equation for the Casimir element alone, and one can show ([**18**], Theorem 5.8) that those coefficients are rational functions in λ. Moreover, unlike the case of matrix coefficients, the recursive equations can be solved explicitly at least for groups of low rank, and an unexpected cancellation keeps the degree of the denominators bounded.

THEOREM 8.1. ([**17**], p. 71, 75) *If P is a maximal parabolic subgroup and T a maximal torus of its Levi component M, then*

$$\Psi_M^{P,\lambda}(t) = t^\lambda \sum_\alpha |\eta_M^G(\check\alpha)| b(-\lambda(\check\alpha), t^{-\alpha}),$$

where the sum is taken over the roots α of $T_{\mathbb{C}}$ such that $\mathfrak{g}_\alpha \subset \mathfrak{n}_{\mathbb{C}}$ but $\mathfrak{g}_\alpha \not\subset \mathfrak{m}_{\mathbb{C}}$, where η_M^G is interpreted as in (6.3) and extended to $\mathfrak{t}_{\mathbb{C}}$ by $\mathbb{C}$-linearity, and

$$b(s,z) = \sum_{n=1}^\infty \frac{z^n}{n+s} = z\int_0^1 \frac{x^s}{1-zx}\,dx \qquad (|z|<1,\ \mathrm{Re}\, s > -1).$$

The function b is a special case of the confluent hypergeometric function and closely related to the incomplete Beta function. It extends to $z \in \mathbb{C}\setminus[1,\infty)$ and has the noteworthy properties

$$z\frac{d}{dz}\, b(s,z) = \frac{z}{1-z} - sb(s,z), \tag{8.1}$$

$$\lim_{z\to 1}\,(b(s_1,z) - b(s_2,z)) = \psi(1+s_2) + \psi(1+s_1), \tag{8.2}$$

$$b(s,-z) - b(-s,-z) = \frac{\pi z^{-s}}{\sin \pi s} - \frac{1}{s}, \qquad z \notin (-\infty, 0], \tag{8.3}$$

where we use the principal branch of the complex power.

The coefficients of the series $\Psi_M^{P,\lambda}$ are rational functions in λ, thus, in several complex variables when M is not maximal. Although no unique decomposition in partial fractions exists for such functions, experience with groups up to rank three suggests that the coefficients should have a canonical expression in partial fractions associated to certain combinatorial data in the root system, which were called root cones in [**18**], p. 794. E. g., in the case of a minimal Levi subgroup of GL(3), there are four root cones in a given Borel subgroup, which give rise to the four terms in the following formula.

THEOREM 8.2. *If $G = \mathrm{GL}(3,\mathbb{R})$, P is the subgroup of upper triangular matrices and $M = A$ the subgroup of diagonal matrices $a = \mathrm{diag}(a_1,a_2,a_3)$, then for each unitary character χ of A with differential $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we have*

$$\begin{aligned}\Psi_A^{P,\chi}(a) = n_A^G \chi(a)\big(&\tilde b(\lambda_{32},\lambda_{31},a_{32},a_{21}) + \tilde b(\lambda_{21},\lambda_{31},a_{21},a_{32}) \\ &+ b(\lambda_{21},a_{31})b(\lambda_{32},a_{32}) + b(\lambda_{21},a_{21})b(\lambda_{32},a_{31})\big),\end{aligned}$$

where

$$\tilde{b}(s_1, s_2, z_1, z_2) = \sum_{n_2=1}^{\infty} \sum_{n_1=n_2}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{(s_1+n_1)(s_2+n_2)}$$
$$= z_1 z_2 \int_0^1 \int_0^1 \frac{x_1^{s_1} x_2^{s_2}}{(1-z_1x_1)(1-z_1z_2x_1x_2)} \, dx_1 \, dx_2$$

($|z_i| < 1$, $\operatorname{Re} s_i > -1$*).*

Here the components λ_i of λ are defined by

$$\lambda(\operatorname{diag}(H_1, H_2, H_3)) = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3,$$

and we use the abbreviations $a_{ij} = a_i/a_j$ and $\lambda_{ij} = \lambda_i - \lambda_j$. The constant n_A^G is the covolume with respect to η_A^G of the lattice generated by the coroots of A.

9. Explicit Fourier transforms for $\mathrm{GL}(\mathbf{2}, \mathbb{R})$

We denote the group $\mathrm{GL}(2, \mathbb{R})$ by M, because it is a factor of a Levi subgroup of $G = \mathrm{GL}(3, \mathbb{R})$ to be considered in the next section. In the group M, we have two conjugacy classes of maximal $\mathbb{R}$-tori represented by the group A of invertible diagonal matrices and the group T of invertible matrices of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. The element $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ serves as a representative for the non-trivial elements of both W_A and W_T, and $c = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & -i \\ -i & 1 \end{smallmatrix}\right)$ defines a Cayley transform $\operatorname{Ad}(c) : \mathfrak{a}_{\mathbb{C}} \to \mathfrak{t}_{\mathbb{C}}$. The roots of A are α_{12} and α_{21}, where

$$\operatorname{diag}(a_1, a_2)^{\alpha_{ij}} = a_{ij} = \frac{a_i}{a_j}.$$

There is only one conjugacy class of Levi $\mathbb{R}$-subgroups also represented by A, and the parabolics with Levi component A are the groups P_{12} and P_{21} of upper resp. lower triangular matrices.

The principal series of M is parametrised by W_A-orbits of unitary characters $\chi \in \Pi_2(A)$. These are of the form

$$\chi(\operatorname{diag}(a_1, a_2)) = \chi_1(a_1)\chi_2(a_2),$$

where

$$\chi_i(a_i) = \chi_i(\operatorname{sgn}(a_i))|a_i|^{\lambda_i}$$

with $\operatorname{Re} \lambda_i = 0$. The unitarily induced representations χ^M are irreducible, the numerators in the character formulas are

$$\Phi^M(\chi^M, a) = \chi(a) + w\chi(a), \qquad \Phi^M(\chi^M, t) = 0$$

for $a \in A$ and $t \in T$, and Langlands' normalising factors for intertwining operators (cf. [**8**], section 3) are

$$r^M_{P_{ji}|P_{ij}}(\chi) = r_{ij}(\chi) = \frac{\Gamma_{\mathbb{R}}(\lambda_{ij} + N)}{\Gamma_{\mathbb{R}}(\lambda_{ij} + N + 1)}.$$

Here

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2}),$$

and $N \in \{0, 1\}$ is determined by $\chi(\gamma) = (-1)^N$, where $\gamma = \operatorname{diag}(-1, -1)$.

It is customary to replace the restriction of $|D|^{1/2}$ to T by

$$\Delta_{ij}(t) = t^{\rho_{ij}} - t^{-\rho_{ij}},$$

where ρ_{ij} is the character of T such that $t^{2\rho_{ij}} = (c^{-1}tc)^{\alpha_{ij}}$. This leads to a modification in the definition of the numerators in character formulas and orbital integrals on orbits of elements of T, and we shall indicate the change by adding a superscript ij.

The discrete series (including limits) of M is parametrised by W_T-orbits of characters $\chi \in \Pi_2(T)$. We have $w\chi = \bar{\chi}$, and the differential $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ of χ must be T-integral, whence the components of $\lambda \circ \mathrm{Ad}(c)$ satisfy

$$\lambda_1 + \bar{\lambda}_2 = 0, \qquad \lambda_{12} = \lambda_1 - \lambda_2 \in \mathbb{Z}.$$

Such a character χ is determined by λ: if $c^{-1}tc = \mathrm{diag}(t_1, t_2)$, then $t_2 = \bar{t}_1$ and

$$\chi(t) = (t_1 t_2)^{\lambda_1} t_2^{\lambda_2 - \lambda_1} = (t_1 t_2)^{\lambda_2} t_1^{\lambda_1 - \lambda_2}. \tag{9.1}$$

Following Harish-Chandra, we define the representation $\sigma_\chi \in \Pi_{\mathrm{temp}}(M)$ by its character values

$$\Theta_{\sigma_\chi}(t) = -\frac{\chi(t) - w\chi(t)}{\Delta_{ij}(t)},$$

thus

$$\Phi^{M,ij}(\sigma_\chi, t) = -(\chi(t) - w\chi(t)),$$

where χ is chosen in its W_T-orbit so that $\lambda_{ij} \geq 0$. The character values on $a = \mathrm{diag}(a_1, a_2) \in A$ are then determined by

$$\Phi^M(\sigma_\chi, a) = 2\,\mathrm{sgn}(a_1) \begin{cases} \chi(cac^{-1}), & \text{if } \mathrm{sgn}(a_1) = \mathrm{sgn}(a_2),\ a_{ij} < 1, \\ w\chi(cac^{-1}), & \text{if } \mathrm{sgn}(a_1) = \mathrm{sgn}(a_2),\ a_{ij} > 1, \\ 0 & \text{if } \mathrm{sgn}(a_1) \neq \mathrm{sgn}(a_2), \end{cases}$$

where χ is defined by (9.1). Those representations σ_χ for which $w\chi \neq \chi$ exhaust the discrete series $\Pi_2(M) = \Pi_{\mathrm{ell}}(M)$, while those with $w\chi = \chi$ belong to the principal series.

For the group M, there is no reducibility under parabolic induction from discrete series, thus $T(M) = \Pi_{\mathrm{temp}}(M)$. The subset $T_{\mathrm{disc}}(M) = \Pi_{\mathrm{disc}}(M)$ is the union of $\Pi_2(M)$ and the set of those χ^M where $r_{ij}(\chi)$ has a pole, i. e., where the restriction of χ to the subgroup $A \cap \mathrm{SL}(2,\mathbb{R})$ is trivial. We shall identify such χ with characters of the quotient torus $A'_M = A/A \cap \mathrm{SL}(2,\mathbb{R})$.

Theorem 9.1. (i) *The Fourier transform of the orbital integral $I_M^M(a)$ with $a \in A \cap M_{\mathrm{reg}}$ is given by*

$$\Phi^M_{M,M}(a, \sigma) = 0, \qquad \Phi^M_{M,A}(a, \chi) = \chi(a) + w\chi(a).$$

(ii) *The Fourier transform of the orbital integral $I_M^{M,ij}(t)$ with $t \in T \cap M_{\mathrm{reg}}$ vanishes on $\Pi_{\mathrm{disc}}(M) \setminus \Pi_2(M)$ and is given by*

$$\Phi^{M,ij}_{M,M}(t, \sigma) = \Phi^{M,ij}(\sigma, t)$$

for $\sigma \in \Pi_2(M)$ and

$$\Phi^{M,ij}_{M,A}(t, \chi) = \frac{n_A^M}{2} \left(\frac{\chi(\gamma)\tilde{\chi}(t^c) - \tilde{\chi}(\gamma t^c)}{\sin \pi\lambda_{ij}} + \frac{w\chi(\gamma) w\tilde{\chi}(t^c) - w\tilde{\chi}(\gamma t^c)}{\sin \pi\lambda_{ji}} \right)$$

for $\chi \in \Pi_2(A)$, where $t^c = c^{-1}tc$ and n_A^M is the length of $\check{\alpha}_{ij}$ with respect to η_A^M. Here we define $\tilde{\chi}$ on a dense subset of $A_{\mathbb{C}}$ by

$$\tilde{\chi}(\mathrm{diag}(a_1, a_2)) = a_1^{\lambda_1} a_2^{\lambda_2}$$

using the principal branches of complex powers.

(iii) *The Fourier transforms of the invariant distribution $I_A^M(a)$ is given for $|a_{ij}| > 1$ by*

$$\Phi_{A,M}^M(a,\sigma) = -\Phi^M(\sigma,a)$$

for $\sigma \in \Pi_2(M)$,

$$\Phi_{A,M}^M(a,\chi^M) = \frac{1}{2}\chi(a)$$

for $\chi \in \Pi_2(A'_M)$,

$$\begin{aligned}\Phi_{A,A}^M(a,\chi) = n_A^M \big(&\chi(a)b(\lambda_{ji},a_{ji}) + \chi(a)u_{ij}(\chi)\\ &+ w\chi(a)b(\lambda_{ij},a_{ji}) + w\chi(a)u_{ij}(w\chi)\big),\end{aligned}$$

where b is the function defined in Theorem 8.1 and $u_{ij}(\chi)$ is the logarithmic derivative of $r_{ij}(\chi)$ with respect to the identification $(\mathfrak{a}_{\mathbb{C}}^M)^* \to \mathbb{C}$ that sends λ to λ_{ij}. If we denote the logarithmic derivative of the Γ-function by ψ, then

$$u_{ij}(\chi) = \frac{1}{2}\,\psi\left(\frac{\lambda_{ij}+N}{2}\right) - \frac{1}{2}\,\psi\left(\frac{\lambda_{ij}+N+1}{2}\right).$$

Assertion (i) is a contained in Example 3.1, assertion (ii) is a special case of the results of [**15**]. Note that assertion (iii) gives the full Fourier transform because $I_A^M(wa) = I_A^M(a)$. The contribution from $\Pi_2(M)$ was given in Theorem 5.3, while the statement about the principal series for $I_A^M(a)$ follows from Theorem 7.2. The contribution of $\chi^M \in \Pi_{\mathrm{disc}}(M) \setminus \Pi_2(M)$ can be read off from [**17**], equation (8), or from [**10**], equation (4.7).

A different proof requires to check the jump relations, as it was done for $\mathrm{SL}(2,\mathbb{R})$ in Lemma 6 of [**17**]. For $\mathrm{GL}(2,\mathbb{R})$, the group A has components (namely with $a_{12} < 0$) where the roots never take the value 1, and one cannot use jump relations to show that the two formulas define a smooth function across $|a_{12}| = 1$. Instead, this can be deduced from the identities (8.3) and

$$\left(\psi\left(\tfrac{s+N+1}{2}\right) - \psi\left(\tfrac{s+N}{2}\right)\right) - \left(\psi\left(\tfrac{-s+N+1}{2}\right) - \psi\left(\tfrac{-s+N}{2}\right)\right) = 2\left(\tfrac{\pi(-1)^N}{\sin\pi s} + \tfrac{1}{s}\right),$$

where $N \in \{0,1\}$. Indeed, it follows that for $x > 0$ the value of

$$\begin{aligned}x^s &\left(b(-s,-x^{-2}) + \tfrac{1}{2}\psi\left(\tfrac{s+N}{2}\right) - \tfrac{1}{2}\psi\left(\tfrac{s+N+1}{2}\right)\right)\\ &+ (-1)^N x^{-s}\left(b(s,-x^{-2}) + \tfrac{1}{2}\psi\left(\tfrac{-s+N}{2}\right) - \tfrac{1}{2}\psi\left(\tfrac{-s+N+1}{2}\right)\right)\end{aligned}$$

does not change if we interchange x^s with $(-1)^N x^{-s}$ and x^{-2} with x^2.

The Fourier transform of $I_A(a)$ defined in (1.3) for singular values of $a \in A$ can also be calculated using equation (8.2) together with the fact that $b(0,z) = -\log(1-z)$, cf. [**17**], section 6.

10. Explicit Fourier transforms for $\mathrm{GL}(3,\mathbb{R})$

In contrast to the preceding section, we shall now use the letter A to denote the group of diagonal matrices $\mathrm{diag}(a_1,a_2,a_3)$, which is a split torus as well as a minimal Levi subgroup in $G = \mathrm{GL}(3,\mathbb{R})$. There are three maximal Levi subgroups M_1, M_2 and M_3 containing A. Namely, M_k consists of the matrices with zero non-diagonal entries in the k-th row and the k-th column. Each of those groups is the direct product of its counterpart in the previous section with the group $\mathrm{GL}(1,\mathbb{R})$. In the same way we obtain from the maximal torus T three maximal tori $T_k \subset M_k$.

Not only minimal, but also maximal Levi subgroups as well as nonsplit tori are all conjugate in G.

The results of the previous section carry over easily to the groups M_k, if we use indices $\{i,j,k\} = \{1,2,3\}$. The tempered representations of M_k are obtained from those of M by multiplying with $\chi_k(m_k)$, where m_k is the kth diagonal entry of $m \in M_k$ and χ_k is a unitary character of $\mathrm{GL}(1,\mathbb{R})$. The same is true of the characters of A and of T_k, of the formulas for the numerators Φ^{M_k} of the distributional characters and, finally, the components of the Fourier transforms. If we denote a character of A or T_k by χ and its differential by λ, then the formulas in Theorem 9.1 remain literally true. Of course, we should use notations w_k, c_k and γ_k to indicate at which diagonal place w, c resp. γ have been augmented by 1, and N should be replaced by $N_k = \chi(\gamma_k)$.

The set $\Pi_{\mathrm{temp}}(G)$ consists of the representations χ^G (with $\chi \in \Pi_2(A)/W_A$) and σ^G (with $\sigma \in \Pi_2(M_k)$, the group W_{M_k} being trivial). There is no reducibility, so $T(G) = \Pi_{\mathrm{temp}}(G)$. The subset $\Pi_{\mathrm{disc}}(G)$ consists of the representations χ^G with $\chi \in \Pi_2(A)$ trivial on $A \cap \mathrm{SL}(3,\mathbb{R})$. Such χ will be identified with characters of the quotient torus $A'_G = A/A \cap \mathrm{SL}(3,\mathbb{R})$.

In order to make statements about arbitrary parabolic subgroups, we use indices $\{i,j,k\} = \{1,2,3\}$. There are two parabolics with Levi component M_k, namely $P_k^{\pm} = M_k N_k^{\pm}$, where the roots of A in $\mathfrak{n}_k^+$ are α_{ik} and α_{jk}, while those in $\mathfrak{n}_k^-$ are α_{ki} and α_{kj}. We denote the restriction of the roots of A in $\mathfrak{n}_k^{\pm}$ to A_{M_k} by $\pm\beta_k$. There are six parabolics with Levi component A, namely $P_{ijk} = AN_{ijk}$ for all permutations (i,j,k) of $(1,2,3)$, where the roots of A in $\mathfrak{n}_{ijk}$ are α_{ij}, α_{jk} and α_{ik}. Langlands' normalising factors for intertwining operators (cf. [**8**], section 3) are

$$r_{P_k^-|P_k^+}(\sigma_\chi) = \frac{2\pi}{\lambda_{ik}}, \qquad r_{P_k^+|P_k^-}(\sigma_\chi) = \frac{2\pi}{\lambda_{kj}}$$

for characters χ of T_k with differential λ such that $\lambda_{ij} > 0$, and

$$\begin{gathered} r_{P_k^-|P_k^+}(\chi^M) = r_{ik}(\chi) r_{jk}(\chi), \qquad r_{P_k^+|P_k^-}(\chi^M) = r_{ki}(\chi) r_{kj}(\chi), \\ r_{P_{jik}|P_{ijk}}(\chi) = r_{P_{jki}|P_{ikj}}(\chi) = r_{P_{kji}|P_{kij}}(\chi) = r_{ij}(\chi), \\ r_{P_{kji}|P_{ijk}}(\chi) = r_{ij}(\chi) r_{jk}(\chi) r_{ik}(\chi) \end{gathered}$$

for characters χ of A.

In many cases, the Fourier transforms of the invariant distributions on G can be reduced to those on Levi subgroups with the aid of the descent identities (5.2):

Theorem 10.1. *The Fourier transforms of ordinary orbital integrals I_G for $m \in M_k \cap G_{\mathrm{reg}}$ are given by*

$$\Phi_{G,M_k}(m,\sigma) = \Phi^{M_k}_{M_k,M_k}(m,\sigma), \qquad \Phi_{G,A}(m,\chi) = \sum_{w \in W_A^{M_k} \backslash W_A} \Phi^{M_k}_{M_k,A}(m, w\chi)$$

for $\sigma \in \Pi_2(M_k)$ and $\chi \in \Pi(A)$. In particular, for $a \in A \cap G_{\mathrm{reg}}$ and $t \in T_k \cap G_{\mathrm{reg}}$,

$$\Phi_{G,A}(a,\chi) = \sum_{w \in W_A} w\chi(a), \qquad \Phi^{ij}_{G,A}(t,\chi) = \frac{n_A^{M_k}}{2} \sum_{w \in W_A} S^{ij}_{T_k,A}(t, w\chi),$$

where

$$S^{ij}_{T_k,A}(t,\chi) = \frac{\chi(\gamma_k)\tilde{\chi}(t^{c_k}) - \tilde{\chi}(\gamma_k t^{c_k})}{\sin \pi\lambda_{ij}}.$$

The Fourier transforms of the invariant distributions $I_{M_k}(a)$ for $a \in A \cap G_{\mathrm{reg}}$ are given by

$$\begin{aligned}\Phi_{M_k,M_k}(a,\sigma) &= d_A^G(M_k,M_i)\Phi_{A,M_i}^{M_i}(a,w_j\sigma) + d_A^G(M_k,M_j)\Phi_{A,M_j}^{M_j}(a,w_i\sigma),\\ \Phi_{M_k,A}(a,\chi) &= \sum_{w\in W_A^{M_i}\backslash W_A} d_A^G(M_k,M_i)\Phi_{A,A}^{M_i}(a,w\chi)\\ &\quad+ \sum_{w\in W_A^{M_j}\backslash W_A} d_A^G(M_k,M_j)\Phi_{A,A}^{M_j}(a,w\chi)\end{aligned}$$

for $\sigma \in \Pi_2(M_k)$ and $\chi \in \Pi(A)$, and

$$\Phi_{M_k,M_k}(a,\chi^{M_k}) = d_A^G(M_k,M_i)w_j\chi(a) + d_A^G(M_k,M_j)w_i\chi(a)$$

for $\chi \in \Pi_2(A'_{M_k})$, where in the last case the measures on $i\mathfrak{a}_{M_i}^*$, $i\mathfrak{a}_{M_j}^*$ and $i\mathfrak{a}_{M_k}^*$ have to be compatible under the action of W_A.

All of these Fourier transforms vanish on $\Pi_{\mathrm{disc}}(G)$.

If we plug in the formulas from Theorem 9.1, some normalising constants become

$$d_A^G(M_k,M_i)n_A^{M_i} = n_{M_k}^G := |\eta_{M_k}^G(\check{\alpha})|$$

in the notation of Theorem 8.1.

Now we come to the distributions that cannot be reduced by descent.

Theorem 10.2. *The Fourier transform of $I_{M_k}^{ij}(t)$ is given for $t \in T_k \cap G_{\mathrm{reg}}$ with $t^{\beta_k} > 1$ by*

$$\begin{aligned}\Phi_{M_k,M_k}^{ij}(t,\sigma_\chi) &= n_{M_k}^G w_k\chi(t)\left(b(\lambda_{kj},t_{ki}) + b(\lambda_{ki},t_{kj}) + \frac{1}{\lambda_{ki}} + \frac{(-t_{kj})^{\lambda_{ik}}}{\sin\pi\lambda_{ik}}\right)\\ &\quad- n_{M_k}^G\chi(t)\left(b(\lambda_{ki},t_{ki}) + b(\lambda_{kj},t_{kj}) + \frac{1}{\lambda_{ki}} + \frac{(-t_{ki})^{\lambda_{ik}}}{\sin\pi\lambda_{ik}}\right),\\ \Phi_{M_k,A}^{ij}(t,\chi) &= \frac{n_A^G}{2}\sum_{w\in W_A} S_{T_k,A}^{ij}(t,w\chi)\Big(b((w\lambda)_{ki},t_{ki}) + u_{ik}(w\chi)\\ &\qquad + b((w\lambda)_{kj},t_{kj}) + u_{jk}(w\chi)\Big)\end{aligned}$$

and for $t \in T_k \cap G_{\mathrm{reg}}$ with $t^{\beta_k} < 1$ by

$$\begin{aligned}\Phi_{M_k,M_k}^{ij}(t,\sigma_\chi) &= n_{M_k}^G w_k\chi(t)\left(b(\lambda_{jk},t_{ik}) + b(\lambda_{ik},t_{jk}) + \frac{1}{\lambda_{jk}} + \frac{(-t_{ik})^{\lambda_{kj}}}{\sin\pi\lambda_{kj}}\right)\\ &\quad- n_{M_k}^G\chi(t)\left(b(\lambda_{ik},t_{ik}) + b(\lambda_{jk},t_{jk}) + \frac{1}{\lambda_{jk}} + \frac{(-t_{jk})^{\lambda_{kj}}}{\sin\pi\lambda_{kj}}\right),\\ \Phi_{M_k,A}^{ij}(t,\chi) &= \frac{n_A^G}{2}\sum_{w\in W_A} S_{T_k,A}^{ij}(t,w\chi)\Big(b((w\lambda)_{ik},t_{ik}) + u_{ki}(w\chi)\\ &\qquad + b((w\lambda)_{jk},t_{jk}) + u_{kj}(w\chi)\Big),\end{aligned}$$

where n_A^G is as in (8.2) and χ is a unitary character of T resp. A. For all $t \in T \cap G_{\mathrm{reg}}$ and $\chi \in \Pi_2(A'_{M_k})$,

$$\Phi^{ij}_{M_k,M_k}(t,\chi^{M_k}) = -\frac{n^G_{M_k}}{4} \sum_{\substack{w\in W_A\\ wM_kw^{-1}\neq M_k}} S^{ij}_{T_k,A}(t,w\chi).$$

Finally, $\hat{I}^{ij}_{M_k}(t)$ vanishes on $\Pi_{\mathrm{disc}}(G)$.

Sketch of the proof. According to Theorem 6.2, the restriction of the Fourier transform to each conected component $(T_k)_{P_k^\pm} \cap G_{\mathrm{reg}}$ is a linear combination of the standard solutions $\Psi^{M_k,\lambda'}_{M_k}(t) = t^{\lambda'}$ (which are characters of $\tilde{T}_k$) and $\Psi^{P_k^\pm,\lambda'}_{M_k}$ (which are given by Theorem 8.1), where $\lambda' \in \mathfrak{t}_{k,\mathbb{C}}$ is in the $W(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{k,\mathbb{C}})$-orbit of λ resp. $\lambda \circ \mathrm{Ad}(c_k)$. The coefficients $c_G^{P_k^\pm}(\lambda')$ of the latter functions can be read off from Theorem 10.1, and in the case of $\Phi_{M_k,A}$, the normalising constants simplify as $n_A^{M_k} n^G_{M_k} = n^G_A$.

The true Fourier transform is smooth on $T_k \cap G_{\mathrm{reg}}$ by Theorem 5.2 and satisfies the asymptotic formula of Theorem 7.1. In fact, it extends continuously to $\{t \in T_k \mid G_t \subset M_k\}$, where G_t/T_k remains compact. Using (8.3), one can check that the two pieces of the putative Fourier transform also combine to a smooth function on $T_k \cap G_{\mathrm{reg}}$ with a continuous extension as above. Moreover, this function satisfies the same asymptotic formula as the true solution, because $b(s,0) = 0$. (In the case of Φ_{M_k,M_k}, one has also to observe that the differential of $\chi(t)(-t_{ki})^{\lambda_{ik}}$ is $\lambda - w_j\lambda$.)

Now we know that the difference between the true and the putative Fourier transform is given as a linear combination of characters $t^{\lambda'}$ on each connected component of $T_k \cap G_{\mathrm{reg}}$, that it extends continuously with the exception of finitely many points and tends to zero as $t \underset{P_k^\pm}{\to} \infty$. Therefore it must vanish.

The assertions about the contributions from $\Pi_{\mathrm{disc}}(G)$ and $\Pi_{\mathrm{disc}}(M_k) \setminus \Pi_2(M_k)$ follow from [**10**], equation (4.7).

Theorem 10.3. *The Fourier transform of the distribution $I_A(a)$ for $a \in A$ such that $|a_1| < |a_2| < |a_3|$ is given by*

$$\begin{aligned}\Phi_{A,M_k}(a,\sigma_\chi) = -n^G_{M_k}\Bigg(&\Phi^{M_1}(s_1\sigma_\chi,a)\left(b(\lambda_{ik},a_{31}) + b(\lambda_{jk},a_{21}) + \frac{1}{\lambda_{kj}}\right)\\ &+ \Phi^{M_2}(s_2\sigma_\chi,a)\Big(b(\lambda_{ik},a_{32}) + b(\lambda_{kj},a_{21})\Big)\\ &+ \Phi^{M_3}(s_3\sigma_\chi,a)\left(b(\lambda_{ki},a_{32}) + b(\lambda_{kj},a_{31}) + \frac{1}{\lambda_{ki}}\right)\Bigg)\end{aligned}$$

for $\chi \in \Pi_2(T_k)$ with $\lambda_{ij} > 0$, where $s_l \in W_A$ conjugates M_k to M_l;

$$\begin{aligned}\Phi_{A,A}(a,\chi) = \sum_{\chi'\in W_A\chi}\Big(&\Psi_A^{P,\chi'}(a) + n_A^G\big(b(\lambda'_{21},a_{21})(u_{13}(\chi') + u_{23}(\chi'))\\ &+ b(\lambda'_{31},a_{31})(u_{12}(\chi') + u_{23}(\chi'))\\ &+ b(\lambda'_{32},a_{32})(u_{12}(\chi') + u_{13}(\chi'))\\ &+ u_{12}(\chi')u_{13}(\chi') + u_{13}(\chi')u_{23}(\chi') + u_{23}(\chi')u_{12}(\chi')\big)\Big)\end{aligned}$$

for $\chi \in \Pi_2(A)$*, where* $\Psi_A^{P,\chi'}$ *is as in Theorem 8.2;*

$$\Phi_{A,M_k}(a,\chi^{M_k}) = \frac{1}{4} \sum_{w \in W_A} \big(d_A^G(M_k,M_i)\Phi_{A,A}^{M_i}(wa,\chi) + d_A^G(M_k,M_j)\Phi_{A,A}^{M_j}(wa,\chi) \big)$$

for $\chi \in \Pi_2(A'_{M_k})$ *and*

$$\Phi_{A,G}(a,\chi^G) = \chi(a)$$

for $\chi \in \Pi_2(A'_G)$.

The statement gives the full Fourier transform, because $I_A(wa) = I_A(a)$ for $w \in W_A$.

Sketch of the proof. As in the preceding proof, we conclude easily from Theorems 6.2, 8.1 and 10.1 that the putative Fourier transforms are correct on a connected component of the given chamber $A_{P_{123}}$ up to a linear combination of terms $t^{\lambda'}$, where λ' is in the W_A-orbit of λ resp. $\lambda \circ \mathrm{Ad}(c^{-1})$.

The formula for $\Phi_{A,A}$ follows from Theorem 7.2.

In the case of Φ_{A,M_k}, no asymptotic formula is available. Here one has to check that the putative Fourier transform satisfies the jump relations (7.1) connecting it with the function Φ_{M_k,M_k} given in Theorem 10.2. This is a cumbersome calculation using equation (8.1). In the end, one concludes that the difference between putative and real Fourier transform is a linear combination of characters on each connected component of A, and being tempered by [**10**], Theorem 4.1, equation (4.4), it must vanish.

As in the preceding theorem, the assertions about the discrete contributions follow from [**10**], equation (4.7).

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, Germany
E-mail address: hoffmann@math.uni-bielefeld.de