# On the non-semisimple contributions to Selberg zeta functions

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## Introduction

The Selberg zeta function of a locally symmetric space $X$ of rank one encodes the lengths and monodromy maps of closed geodesics. For spaces of finite volume, there is a rich theory including the relation of this zeta function to regularized determinants of Laplacians, which is proved using the trace formula for the isometry group $G$ of the universal covering space of $X$. The case of vector bundles over the unit sphere bundle $S(X)$, which are parametrized by representations $\sigma$ of a proper Levi subgroup $M$, has for a long time been treated only for compact $X$ due to the lack of knowledge of the Fourier transforms of weighted orbital integrals on the generalised principal series. These Fourier transforms have been calculated explicitly in [4], and applications to zeta functions for odd-dimensional hyperbolic manifolds were given in [6] and [7] under the assumption that the only non-semisimple elements of the fundamental group $\Gamma$ of $X$ are the unipotent ones. In that case, the relevant Fourier transform simplified, and one could check that its contribution to the formula for the logarithmic derivative of the Selberg zeta function had integer residues. This is necessary for the proof of the meromorphic continuation of that function.

In the present paper, we present a new formula for those Fourier transforms in terms of characters of several (in fact, infinitely many) representations of $M$ including $\sigma$. This enables us to reformulate the contribution from a cusp to the determinant formula with the help of the trace formula for a discrete subgroup of $M$, which was trivial in the neat case. It turns out the the full contribution from a cusp has at least half-integer residues in general. We restrict ourselves to test functions in the subspace $\mathcal{C}_{\mathrm{con}}(G)$ of the Schwartz space spanned by wave packets. The application to the Selberg zeta function will be given in another paper.

We will mostly apply the notation of our paper [4]. Thus, let $G$ be a connected semisimple Lie group of real rank one contained in its simply connected complexification, and let $K$ be a maximal compact subgroup of $G$. For $g \in G$, we have the Weyl discriminant

$$D^G(g) = \det(\mathrm{Id} - \mathrm{Ad}(g))_{\mathfrak{g}/\mathfrak{g}_g}$$

and the orbital integral

$$J_G(g,f) = |D^G(g)|^{1/2} \int_{G/G_g} f(xgx^{-1})$$

defined for all $f$ in the Schwartz space $\mathcal{C}(G)$. We choose a proper parabolic subgroup with Levi decomposition $P = MN$ and a Cartan subgroup $A$ of $M$. We denote the (one-dimensional) vectorial part of $A$ by $A_R$, so that we have direct product decompositions $A = A_I A_R$ and $M = M_I A_R$, and we fix a positive multiple $\lambda_P$ of the roots of $\mathfrak{a}_R$ in $\mathfrak{n}$ to normalise the Haar measure on $A_R$. Using the map $H_P : G \to \mathfrak{a}$ defined by

$$H_P(kan) = \log a \qquad \text{for } k \in K,\ a \in A_R,\ n \in N$$

and its analogue for the opposite parabolic $\bar{P}$ of $P$ with respect to $M$, one defines the weight factor $v(x) = \lambda_P(H_P(x) - H_{\bar{P}}(x))$ and the weighted orbital integral

$$J_M(m,f) = |D^G(m)|^{1/2} \int_{G/G_m} f(xmx^{-1}) v(x)\, dx$$

for $f \in \mathcal{C}(G)$ and $m \in M$ such that $G_m^0 \subset M$.

We identify $\mathfrak{a}^*_{R,\mathbb{C}}$ with $\mathbb{C}$ so that $\lambda_P$ corresponds to 1 and denote the derivative of a holomorphic function $h$ by $\partial_P h$. The twist of an irreducible representation $\sigma$ of $M$ by $\lambda \in \mathfrak{a}^*_{R,\mathbb{C}}$ is defined as $\sigma_\lambda(ma) = \sigma(ma)a^\lambda$ for $m \in M$, $a \in A_R$. Thereby we identify $\hat{M} = \hat{M}_I \times i\mathfrak{a}^*_R$, and we write the Plancherel measure of $M$ as a contour integral

$$\frac{1}{2\pi i} \int_{\hat{M}} h(\sigma)\, d\sigma = \sum_{\sigma \in \hat{M}_I} \frac{1}{2\pi i} \int_{i\mathfrak{a}^*_R} h(\sigma_\lambda)\, d\lambda.$$

The distributional trace of an admissible representation $\pi$ of $G$ will be denoted by $\Theta_\pi$. The generalised principal series $\pi_{P,\sigma}$ induced from $\sigma$ via $P$ is realised it in the compact picture, a space of functions on $K$ unchanged under twists by $\lambda \in i\mathfrak{a}_R^*$. This makes it possible to define the weighted character

$$J_P(\sigma, f) = -\operatorname{tr}(\pi_{P,\sigma}(f) J_{\bar{P}|P}(\sigma)^{-1} \partial_P J_{\bar{P}|P}(\sigma)),$$

where $J_{\bar{P}|P}(\sigma)$ denotes the Knapp-Stein intertwining operator from $\pi_{P,\sigma}$ to $\pi_{\bar{P},\sigma}$. As in [4], we consider a version of Arthur's invariant tempered distributions $I_M(m, f)$ defined in [1], section 10, viz.

$$I_P(m, f) = J_M(m, f) - |D^M(m)|^{1/2} \frac{1}{2\pi i} \int_{\hat{M}} \Theta_{\check{\sigma}}(m) J_M(\sigma, f)\, d\sigma \\ - |D^M(m)|^{1/2} \sum_{\sigma \in \hat{M}_I} \frac{n(\sigma)}{2} \Theta_{\check{\sigma}}(m) \Theta_{\pi_\sigma}(f),$$

where $2n(\sigma)$ is the order of the zero of the Plancherel density of $G$ at $\sigma$.

# 1 An alternative formula for the Fourier transform

For applications in the trace formula, it is more natural to consider, for $m \in M$ such that $G_m^0 \subset M$, the partially normalised weighted orbital integral

$$\tilde{J}_M(m, f) = |D_M^G(m)|^{1/2} \int_{G/G_m} f(xmx^{-1}) v(x)\, dx,$$

where

$$D_M^G(m) = \det(\mathrm{Id} - \mathrm{Ad}(m))_{\mathfrak{g}/\mathfrak{m}}.$$

We assume that the compact quotient $G_m^0/A_R$ has volume one, so that integrating over $G/A_R$ gives the same result. This shows that $\tilde{J}_M(m, f)$ depends continuously on $m$. It is related to the earlier version as

$$J_M(m, f) = |D^M(m)|^{1/2} \tilde{J}_M(m, f),$$

where

$$D^M(m) = \det(\mathrm{Id} - \mathrm{Ad}(m))_{\mathfrak{m}/\mathfrak{g}_m}.$$

Of course, we can also define the invariant distribution $\tilde{I}_P(m)$ in the same fashion, which is related to our earlier distribution as

$$I_P(m,f) = |D^M(m)|^{1/2}\tilde{I}_P(m,f).$$

Let $A$ be a Cartan subgroup of $M$ and $\Lambda \subset \mathfrak{a}_{\mathbb{C}}^*$ the set of $A$-integral weights. The set of roots of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{n}_{\mathbb{C}}$ is denoted by $\Sigma_P^+$. If $\Sigma$ is a half-system of positive roots of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{m}_{\mathbb{C}}$ and $\lambda \in \Lambda$ the infinitesimal character of a finite-dimensional representation $\sigma$ of $M$, then there exists $w \in W(M,A)$ such that $w\lambda$ is $\Sigma$-dominant, and we set

$$\Theta_{\Sigma,\lambda} = \varepsilon_M(w)\Theta_\sigma,$$

where $\Theta_\sigma(m) = \operatorname{tr}\sigma(m)$ is the character of $\sigma$. If $\lambda \in \Lambda$ is not regular, then we set $\Theta_{\Sigma,\lambda} = 0$.

**Theorem 1** *If $f \in \mathcal{C}_{\text{con}}(G)$ and $m \in M$ such that $G_m^0 \subset M$ and $m^{\lambda_P} \geq 1$, then*

$$\tilde{I}_P(m,f) = \frac{1}{2\pi i}\int_{\hat{M}} \tilde{\Omega}_P(m,\check{\sigma})\Theta_{\pi_\sigma}(f)\, d\sigma$$

*with a function $\tilde{\Omega}_P$ that is continuous in $\sigma$ and given by*

$$\tilde{\Omega}_P(m,\sigma) = \sum_{n=1}^{\infty}\sum_{\alpha\in\Sigma_P^+}\frac{\lambda_P(H_\alpha)}{2}\left(\frac{\Theta_{\Sigma,\lambda-n\alpha}(m)}{n-\lambda(H_\alpha)} + \frac{\Theta_{\Sigma,w_0\lambda-n\alpha}(m)}{n-w_0\lambda(H_\alpha)}\right),$$

*where $\lambda \in \Lambda$ is the $\Sigma$-dominant infinitesimal character of $\sigma$.*

The convergence of the series should be uniform on compact subsets of its domain of definition, but the necessary estimates seem too cumbersome to be carried out in general. On the subset where $m^{\lambda_P} > 1$, the convergence is easily seen to be absolutely uniform on compact subsets.

*Proof.* Since both sides are class functions of $m$, we can assume that $m = a \in A$, where the condition $G_a^0 \subset M$ now means that $a \in A''$ in the notation of [4]. In Theorem 1 of that paper we had also considered, for $a \in A'$, the distribution

$$I_{P,\Sigma}(a,f) = \Delta_\Sigma(a)\tilde{I}_P(a,f)$$

and computed its Fourier transform in terms of a function

$$\Omega_{P,\Sigma}(a,\sigma) = \Delta_\Sigma(a)\tilde{\Omega}_P(a,\sigma),$$

where $\Delta_\Sigma$ is the Weyl denominator for $\Sigma$. Recall that for $\sigma|_{A_R} \neq 1$ with $\Sigma$-dominant infinitesimal character $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and for $a^{\lambda_P} > 1$ we had proved a formula

$$\Omega_{P,\Sigma}(a,\sigma) = \frac{1}{2} \sum_{w \in W(G,A)} \varepsilon_M(w) a^{w\lambda} \sum_{\alpha \in \Sigma_P^+} \lambda_P(H_\alpha) \sum_{n=1}^{\infty} \frac{a^{-n\alpha}}{n - w\lambda(H_\alpha)}.$$

We had also seen by Abel summation that the series is uniformly convergent with all invariant derivatives, hence smooth on $\{a \in A'' \mid a^{\lambda_P} \geq 1\}$.

Using Lemma 1 of [4], we split the sum over $W(G,A)$ into a sum over $W(G,A_R) \cong \{1, w_0\}$ and a sum over $w \in W(M,A)$. Then we substitute $w\alpha$ for $\alpha$ and bring the summation over $w$ innermost. The interior sum can be expressed in terms of the function

$$\Phi_\nu(a) = \sum_{w \in W(M,A)} \varepsilon_M(w) a^{w\nu}.$$

Assuming that $a \in A'$ and dividing by $\Delta_\Sigma(a)$, we obtain our formula for $\tilde{\Omega}_P$ with the help of the identity

$$\Theta_{\Sigma,\nu}(a) = \Delta_\Sigma(a)^{-1} \Phi_\nu(a),$$

which is Weyl's character formula for regular $\nu$ while both sides vanish for singular $\nu$.

For general $a \in A''$ such that $a^{\lambda_P} \geq 1$, we have

$$\tilde{I}_P(a,f) = \lim_{a' \to a} \frac{I_{P,\Sigma}(a',f)}{\Delta_\Sigma(a')} = \frac{D_{\Pi_a} I_{P,\Sigma}(a,f)}{D_{\Pi_a} \Delta_\Sigma(a)},$$

where $a'$ stays in $A'$, $\Pi_a \in S(\mathfrak{a}_{I,\mathbb{C}})$ is the product of the coroots $H_\alpha$ over all roots in the set $\Sigma_a = \{\alpha \in \Sigma \mid a^\alpha = 1\}$, and $D_{\Pi_a}$ is the corresponding differential operator on $A$ that is applied before evaluating at $a$. Our formula will follow from the identity

$$\Theta_{\Sigma,\nu}(a) = \lim_{a' \to a} \frac{\Phi_\nu(a')}{\Delta_\Sigma(a')} = \frac{D_{\Pi_a} \Phi_\nu(a)}{D_{\Pi_a} \Delta_\Sigma(a)} \quad (1)$$

if we can show that

$$C_{M_a} \Omega_{P,\Sigma}(a,\sigma) = D_{\Pi_a} \Omega_{P,\Sigma}(a,\sigma)$$

exists in the space of tempered distributions on $\hat{M}$ and can be obtained by differentiating and passing to the limit termwise.

This is obvious for $a^{\lambda_P} > 1$ as the series is then absolutely uniformly convergent on compact subsets together with all derivatives. In Theorem 5 of [4] an alternative formula was found, viz.

$$\sum_{w\in W(M,A)} \varepsilon_M(w) \sum_{\alpha\in\Sigma_P^+} \frac{\Pi_a(w(\lambda-n\alpha))a^{w(\lambda-n\alpha)}}{n-\lambda(H_\alpha)}$$

$$= \sum_{w\in W(M,A)} \varepsilon_M(w) \sum_{\alpha\in\Sigma_P^+} \frac{\Pi_a(s_\alpha w\lambda)a^{w\lambda-n\alpha}}{n-w\lambda(H_\alpha)}. \quad (2)$$

Actually, only the equality of the series obtained by summing either side over all natural numbers $n$ was proved, but that is a power series in the $A_R$-component of $a$, whose terms are uniquely determined. The alternative series was shown to converge for all $a \in A''$ with $a^{\lambda_P} \geq 1$. Since the equality of terms extends to the larger domain, our series is convergent there, too. □

When $m \in M$ approaches the set where $G_m^0 \not\subset M$ then $\tilde{I}_P(m,f)$ blows up with leading term given by the ordinary orbital integral. Here, we need a partially normalised version of the latter, viz.

$$\tilde{J}_G(m,f) = |D_M^G(m)|^{1/2} \int_{G/G_m} f(xmx^{-1})\,dx$$

for $m \in M$ such that $G_m^0 \subset M$, of course with the same normalisation of Haar measure as for $\tilde{J}_M(m,f)$. Now for elements $m \in M$ such that $G_m^0 \not\subset M$, let $\beta_m$ be the only reduced root of $(\mathfrak{g}_m, \mathfrak{a}_R)$ positive with respect to $P$ and denote the corresponding coroot by $H_{\beta_m} \in \mathfrak{a}_R$. One defines

$$\tilde{I}_P(m,f) = \lim_{a_R\to 1} \left(\tilde{I}_P(ma_R,f) + \lambda_P(H_{\beta_m}) \log\left|a_R^{\beta_m/2} - a_R^{-\beta_m/2}\right| \cdot \tilde{J}_G(ma_R,f)\right),$$

where $a_R$ runs through the nontrivial elements of $A_R$. We have the following analogue of Theorem 6 of [4].

**Theorem 2** *If $f \in \mathcal{C}_{\mathrm{con}}(G)$ and $m \in M$ such that $G_m^0 \not\subset M$, then*

$$\tilde{I}_P(m,f) = \frac{1}{2\pi i} \int_{\hat{M}} \tilde{\Omega}_P(m,\check{\sigma})\Theta_{\pi_\sigma}(f)\,d\sigma$$

*with $\tilde{\Omega}_P(m,\sigma)$ given by*

$$\frac{1}{2}\sum_{w\in\{1,w_0\}}\left(\sum_{n=1}^{\infty}\left(\sum_{\alpha\in\Sigma_P^+}\lambda_P(H_\alpha)\frac{\Theta_{\Sigma,w\lambda-n\alpha}(m)}{n-w\lambda(H_\alpha)}-\lambda_P(H_{\beta_m})\frac{\Theta_{\Sigma,w\lambda}(m)}{n}\right)\right.$$
$$\left.-\lambda_P(H_\beta)\Theta_{\Sigma,w\lambda}(m)\log\frac{\beta(H_{\beta_m})}{2}\right),$$

*where $\lambda\in\Lambda$ is the $\Sigma$-dominant infinitesimal character of $\sigma$, and the term containing the real root $\beta$ has to be omitted if that root does not exist.*

*Proof.* Again we may assume that $m=a\in A$, and the condition $G_a^0\not\subset M$ implies $a\in A_I$. Since our distribution is related to its analogue considered in [4] as

$$\tilde{I}_P(a,f)=\frac{C_{M_a}I_P(a,f)}{D_{\Pi_a}\Delta_\Sigma(a)},$$

we can obtain its Fourier transform as

$$\tilde{\Omega}_P(a,\sigma)=\frac{C_{M_a}\Omega_P(a,\sigma)}{D_{\Pi_a}\Delta_\Sigma(a)}.$$

In Theorem 6 of [4] we have already computed

$$C_{M_a}\Omega_P(a,\sigma)=\frac{1}{2}\sum_{w\in W(G,A)}\varepsilon_M(w)a^{w\lambda}\sum_{\alpha\in\Sigma_P^+}\Pi_a(s_\alpha w\lambda)\phi_\alpha(a,w\lambda),$$

where

$$\phi_\alpha(a,\lambda)=\lambda_P(H_\alpha)\sum_{n=1}^{\infty}\frac{a^{-n\alpha}}{n-\lambda(H_\alpha)}$$

if $a^\alpha\neq 1$, while

$$\phi_\alpha(a,\lambda)=\lambda_P(H_\alpha)\left(\sum_{n=1}^{\infty}\left(\frac{1}{n-\lambda(H_\alpha)}-\frac{1}{n}\right)-\log\frac{\alpha(H_{\beta_a})}{2}\right)$$

if $a^\alpha=1$.

The terms common to both cases can be easily combined into a sum over $\Sigma_P^+$. In order to treat the additional contribution from $\Sigma_{P,a}^+ = \{\alpha \in \Sigma_P^+ \mid a^\alpha = 1\}$, we use the identity

$$\sum_{w\in W(G,A)} \varepsilon_M(w) \left( \lambda_P(H_{\beta_a})\Pi_a(w\lambda) - \sum_{\alpha\in\Sigma_{P,a}^+} \lambda_P(H_\alpha)\Pi_a(s_\alpha w\lambda) \right) = 0$$

proved in [4], p. 95. The term $\log \frac{\alpha(H_{\beta_a})}{2}$ is nonzero (in fact, equal to $\log 2$) only for the roots from the subset $\Sigma_{P_\gamma,a}^+ = \{\alpha \in \Sigma_P^+ \mid a^\alpha = 1,\ \gamma^\alpha = 1\}$, where $\gamma = \exp 2\pi H_\beta$ is the element in the centre of $M_I$ introduced in [3], §24. In the analogue of the above identity

$$\sum_{w\in W(G_\gamma,A)} \varepsilon_M(w) \left( \lambda_P(H_\beta)\Pi_a(w\lambda) - \sum_{\alpha\in\Sigma_{P_\gamma,a}^+} \lambda_P(H_\alpha)\Pi_a(s_\alpha w\lambda) \right) = 0$$

for the group $G_\gamma$ we can enlarge the summation to $W(G,A)$. Now we get

$$C_{M_a}\Omega_P(a,\sigma) = \frac{1}{2} \sum_{w\in W(G,A)} \varepsilon_M(w) \Bigg( -\lambda_P(H_\beta) \log \frac{\beta(H_{\beta_a})}{2}\, \Pi_a(w\lambda)a^{w\lambda} + \sum_{n=1}^{\infty} \left( \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha) \frac{\Pi_a(s_\alpha w\lambda)a^{w\lambda-n\alpha}}{n - w\lambda(H_\alpha)} - \lambda_P(H_{\beta_a}) \frac{\Pi_a(w\lambda)a^{w\lambda}}{n} \right) \Bigg),$$

where the term containing $\beta$ has to be omitted if there is no real root $\beta$ in $\Sigma_P^+$. As in the preceding proof, we split the sum over $W(G,A)$, substitute $w\alpha$ for $\alpha$ and bring the summation over $W(M,A)$ innermost. Rewriting the terms with the aid of the identity (2), we obtain

$$C_{M_a}\Omega_P(a,\sigma) = \frac{1}{2} \sum_{w\in\{1,w_0\}} \Bigg( -\lambda_P(H_\beta) \log \frac{\beta(H_{\beta_a})}{2}\, D_{\Pi_a}\Phi_{w\lambda}(a) + \sum_{n=1}^{\infty} \left( \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha) \frac{D_{\Pi_a}\Phi_{w\lambda-n\alpha}(a)}{n - w\lambda(H_\alpha)} - \lambda_P(H_{\beta_a}) \frac{D_{\Pi_a}\Phi_{w\lambda}(a)}{n} \right) \Bigg).$$

It remains to divide by $D_{\Pi_a}\Delta_\Sigma(a)$ and use formula (1). $\square$

# 2 The non-semisimple contribution to the trace formula

In the trace formula, weighted orbital integrals appear in the cuspidal terms on the geometric side. In the invariant trace formula, they are replaced by the invariant distributions $I_P$ or their analogues $I_M$ (see [5]). We will see that for groups $G$ of real rank one the total contribution from a cusp simplifies if we use the alternative formulae from Theorems 1 and 2.

If $\Gamma$ is a lattice in $G$ and $P$ a cuspidal parabolic subgroup with Levi decomposition $P = MN$, we denote the projection of $\Gamma \cap P$ to $M$ by $\Gamma_P$. For $\lambda \in \Lambda$ there is $w \in W(M, A)$ such that $w\lambda$ is the $\Sigma$-dominant infinitesimal character of a representation $\sigma$ of $M$, and we set

$$d_{\Gamma,\Sigma}(\lambda) = \varepsilon_M(w) \dim V_\sigma^{\Gamma_P}.$$

**Theorem 3** *If $f \in \mathcal{C}_{\mathrm{con}}(G)$, $\Gamma$ is a lattice in $G$ and $P = MN$ is a cuspidal subgroup, then*

$$\sum_{[\delta] \subset \Gamma_P} |\Gamma_{P,\delta}|^{-1} \tilde{I}_P(\delta, f) = \int_{\hat{M}} \tilde{\Omega}_{P,\Gamma}(\check{\sigma}) \Theta_{\pi_\sigma}(f) \, d\sigma,$$

*where $[\delta]$ runs through the conjugacy classes in $\Gamma_P$ and $\tilde{\Omega}_{P,\Gamma}(\sigma)$ equals*

$$\frac{1}{2} \sum_{w \in \{1, w_0\}} \left( \sum_{n=1}^{\infty} \left( \sum_{\alpha \in \Sigma_P^+} \lambda_P(H_\alpha) \frac{d_{\Gamma,\Sigma}(w\lambda - n\alpha)}{n - w\lambda(H_\alpha)} - \frac{d'_{\Gamma,\Sigma}(w\lambda)}{n} \right) - d''_{\Gamma,\Sigma}(w\lambda) \right).$$

*Here $\lambda$ is the $\Sigma$-dominant infinitesimal character of $\sigma$ and*

$$d'_{\Gamma,\Sigma}(\lambda) = \sum_{\substack{[\delta] \subset \Gamma_P \\ G_\delta \not\subset M}} |\Gamma_{P,\delta}|^{-1} \lambda_P(H_{\beta_\delta}) \Theta_\lambda(\delta),$$

$$d''_{\Gamma,\Sigma}(\lambda) = \sum_{\substack{[\delta] \subset \Gamma_P \\ G_\delta \not\subset M}} |\Gamma_{P,\delta}|^{-1} \lambda_P(H_{\beta_\delta}) \Theta_\lambda(\delta) \log \frac{\beta(H_{\beta_\delta})}{2}.$$

*Proof.* For fixed $\sigma \in \hat{M}_I$ realised on a finite-dimensional vector space $V_\sigma$, the space of the induced representation $\pi = \mathrm{Ind}_{\Gamma_P}^{M_I}(\sigma)$ consists of all classes of measurable functions $\phi : M_I \to V_\sigma$ such that $\phi(\delta m) = \sigma(\delta)\phi(m)$ for all $\delta \in \Gamma_P$ and $m \in M_I$. As usual, one proves that for $h \in L^1(M_I)$ we have

$$\pi(h)\phi(m) = \int_{\Gamma_P \backslash M_I} K(m, m')\phi(m')\, dm,$$

where

$$K(m, m') = \sum_{\delta \in \Gamma_P} h(m^{-1}\delta m')\sigma(\delta).$$

We have the trace formula

$$\operatorname{tr} \pi(h) = \int_{\Gamma_P \backslash M_I} \operatorname{tr} K(m, m)\, dm.$$

For $h$ equal to the constant 1, we get the projection $\pi(1)$ onto the $M_I$-invariants, and this trace formula becomes

$$[\pi : \mathbf{1}_{M_I}] = \sum_{[\delta] \subset \Gamma_P} |\Gamma_{P,\delta}|^{-1} \Theta_\sigma(\delta).$$

By Frobenius reciprocity, the left-hand side equals

$$[\sigma_{\Gamma_P} : \mathbf{1}_{\Gamma_P}] = \dim V_\sigma^{\Gamma_P}.$$

In particular, we get

$$d_{\Gamma,\Sigma}(\lambda) = \sum_{[\delta] \subset \Gamma_P} |\Gamma_{P,\delta}|^{-1} \Theta_\lambda(\delta).$$

Together with the definitions of $d'_{\Gamma,\Sigma}(\lambda)$ and $d''_{\Gamma,\Sigma}(\lambda)$, this allows us to write the result we get from Theorems 1 and 2 in the asserted form. □

# 3 The non-semisimple contribution to the Selberg zeta function

The Selberg zeta function encodes the lengths of closed geodesics on the locally symmetric space $X/\Gamma$, where $X$ is a symmetric space on which $G$ acts transitively from the right. One may also include the monodromy representations in an equivariant vector bundle over the unit tangent bundle of $X$, pushed down to $X/\Gamma$. There is a $G$-invariant symmetric bilinear form on $\mathfrak{g}$ such that the Riemannian metric on the tangent space at $x$ is identified with the restriction of that form to the orthogonal complement of the stabiliser of $x$ in $\mathfrak{g}$. We assume that the value of that form on $H \in \mathfrak{a}_R$ is $\lambda_P(H)^2$.

The Selberg zeta function $Z_\Gamma(\sigma, s)$ depends on a representation of $M_I$, which encodes the vector bundle, and a complex number $s$ with $\operatorname{Re} s \gg 0$. We assume that the representation is $\bigoplus_{w \in W(G,A_R)} w\sigma$ (consisting of one or two summands) and set $\sigma_s(ma_R) = \sigma(m)a^{s\lambda_P}$. The logarithmic derivative of $Z_\Gamma(\sigma, s)$ considered as a function of $s^2$, viz.

$$\frac{1}{2s} \cdot \frac{Z'_\Gamma(\sigma, s)}{Z_\Gamma(\sigma, s)},$$

equals the hyperbolic contribution to the trace formula for the test function $f_s \in \mathcal{C}_{\mathrm{con}}(G)$ such that

$$\Theta_{\pi_{\sigma_{s'}}}(f_s) = \Theta_{\pi_{w_0\sigma_{s'}}}(f_s) = (s^2 - s'^2)^{-1}$$

while $\Theta_{\pi_{\sigma'_{s'}}}(f_s) = 0$ for $\sigma' \in \hat{M}_I$ different from (the classes of) $\sigma$ and $w_0\sigma$.

The full trace formula converges only after some regularisation. We choose the following version (cf. [2]). For a finite subset $S$ of the complex plane and any meromorphic function $h_s$, write

$$[h(s)]_{s\in S} = \sum_{s\in S} h(s) \prod_{\substack{s''\in S\\ s''\neq s}} (s''^2 - s^2)^{-1},$$

which depends linearly on $h$. The version of the resolvent identity

$$[(s^2 - s'^2)^{-1}]_{s\in S} = \prod_{s\in S} (s^2 - s'^2)^{-1},$$

which is valid for $s' \notin S$, shows that

$$\Theta_{\pi_{\sigma_{s'}}}([f_s]_{s\in S}) = O\left(|s'|^{-2\#(S)}\right)$$

as $|s'| \to \infty$.

**Theorem 4** *If $S$ is a finite subset of the open right complex half-plane with sufficiently many elements, then*

$$\sum_{[\delta]\subset\Gamma_P} |\Gamma_{P,\delta}|^{-1} \tilde{I}_P(\delta, [f_s]_{s\in S}) = \sum_{\sigma'\in\{\sigma, w_0\sigma\}} \left[\frac{1}{2s} \tilde{\Omega}^+_{P,\Gamma}(\check{\sigma}'_s)\right]_{s\in S},$$

*where*

$$\tilde{\Omega}^{+}_{P,\Gamma}(\sigma_s) = \sum_{n=1}^{\infty} \left( {\sum_{\alpha\in\Sigma_P^+}}' \frac{\lambda_P(H_\alpha) d_{\Gamma,\Sigma}(\mu - n\alpha)}{n - (\pm s\lambda_P + \mu)(H_\alpha)} - \frac{d'_{\Gamma,\Sigma}(\mu)}{n} \right) - d''_{\Gamma,\Sigma}(\mu).$$

*Here $\mu$ is the $\Sigma$-dominant infinitesimal character of $\sigma$, and the sign has to be chosen so that the denominator is holomorphic for* $\operatorname{Re} s > 0$. *The dash at the summation sign means that the terms with denominators vanishing at $s = 0$ have to be omitted.*

*Proof.* First we show that the terms with vanishing denominators could have been omitted already in Theorems 1, 2 and 3. Indeed, let us plug in a representation $\sigma_s$, whose $\Sigma$-dominant infinitesimal character is $s\lambda_P + \mu$. Since the weighted character $J_M(\sigma_s, f)$ is regular at $s = 0$, the unnormalised version $J_P(\sigma_s, f)$ has a pole at $s = 0$ if and only if the Plancherel density has a zero there, in which case $w_0\sigma = \sigma$. Then the averaging over $w \in \{1, w_0\}$ makes $\tilde{\Omega}_P(m, \sigma_s)$ symmetric in $s$ and removes the pole. Thus its residue, which is just the coefficient of $\frac{1}{s}$ in the series, vanishes.

Now we come to the proof of the asserted formula. According to Theorem 3, the left-hand side equals

$$\sum_{\sigma' \in \{\sigma, w_0\sigma\}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left[ \frac{\tilde{\Omega}_{P,\Gamma}(\check{\sigma}'_{s'})}{s^2 - s'^2} \right]_{s\in S} ds'.$$

Here we plug in the formula for $\tilde{\Omega}_{P,\Gamma}(\sigma_{s'})$ given in that Theorem. Since $\Theta_{\pi_{w_0\sigma}} = \Theta_{\pi_\sigma}$, we may omit the averaging over $w \in \{1, w_0\}$. Then apply the following fact.

Let $h$ be a function which is holomorphic on a neighbourhood of either the left or right complex half-plane and satisfies $h(s) = O(|s|^c)$ with $c < 2\#(S) - 1$, where $S$ is as in the theorem. Then

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left[ \frac{h(s')}{s^2 - s'^2} \right]_{s\in S} ds' = \left[ \frac{h(\pm s)}{2s} \right]_{s\in S},$$

where we have to choose the sign $+$ if $h$ is holomorphic in the right half-plane and the sign $-$ if $h$ is holomorphic in the left half-plane. This follows from the residue theorem applied to a half-disc about 0 with radius tending to infinity.

In our series, we have to isolate the first few terms so that the rest is holomorphic in one of the half-planes. □

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