Ringel-Hall Algebras

Andrew W. Hubery

1. Introduction

Two of the first results learned in a linear algebra course concern finding normal forms for linear maps between vector spaces.

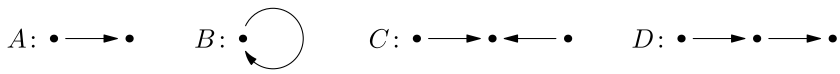
We may consider a linear map $f\colon V \to W$ between vector spaces over some field k. There exist isomorphisms $V \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$ and $W \cong \mathrm{Im}(f) \oplus \mathrm{Coker}(f)$, and with respect to these decompositions $f = \mathrm{id} \oplus 0$.

If we consider a linear map $f\colon V \to V$ with V finite dimensional, then we can express f as a direct sum of Jordan blocks, and each Jordan block is determined by a monic irreducible polynomial in $k[T]$ together with a positive integer.

We can generalise such problems to arbitrary configurations of vector spaces and linear maps. For example

$$U_1 \xrightarrow{f} V \xleftarrow{g} U_2 \quad \text{or} \quad U \xrightarrow{f} V \xrightarrow{g} W.$$

We represent such problems diagrammatically by drawing a dot for each vector space and an arrow for each linear map. So, the four problems listed above correspond to the four diagrams



Such a diagram is called a quiver, and a configuration of vector spaces and linear maps is called a representation of the quiver: that is, we take a vector space for each vertex and a linear map for each arrow.

Given two representations of the same quiver, we define the direct sum to be the representation produced by taking the direct sums of the vector spaces and the linear maps for each vertex and each arrow. Recall that if $f\colon U \to V$ and $f'\colon U' \to V'$ are linear maps, then the direct sum is the linear map

$$f \oplus f'\colon U \oplus U' \to V \oplus V', \quad (u, u') \mapsto (f(u), f'(u')).$$

If we choose bases, then the direct sum corresponds to a block diagonal matrix. A representation is called decomposable if there exists a choice of basis for each vector space such that all linear maps are simultaneously represented by block diagonal matrices. If there exists no such choice of bases, then the representation is called indecomposable. Clearly each finite dimensional representation is, after base change, the direct sum of indecomposable representations, and the Krull-Remak-Schmidt Theorem states that this decomposition is essentially unique.

The basic aim is therefore to classify all possible indecomposable representations (up to base change) of a given quiver.

For the one-subspace quiver, labelled A above, we have seen that every representation can be written as the direct sum of the three indecomposable representations

$$k \xrightarrow{1} k \qquad k \longrightarrow 0 \qquad 0 \longrightarrow k$$

For the Jordan quiver, quiver B, each representation is a direct sum of Jordan blocks, and these are indecomposable as seen by considering the corresponding minimal polynomials.

For the two-subspace quiver, quiver C, we can use the rank-nullity theorem to show that there are precisely six indecomposable representations

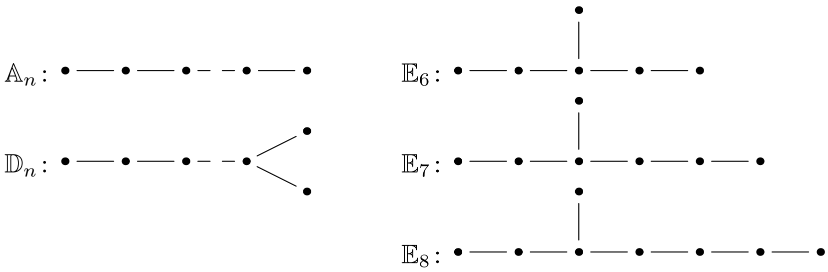
$$\begin{array}{ccc} k \xrightarrow{1} k \xleftarrow{1} k & k \xrightarrow{1} k \longleftarrow 0 & 0 \longrightarrow k \xleftarrow{1} k \\ 0 \longrightarrow k \longleftarrow 0 & k \longrightarrow 0 \longleftarrow 0 & 0 \longrightarrow 0 \longleftarrow k \end{array}$$

The problem of classifying all indecomposable representations of a given quiver is generally consider impossible, so one may ask for which quivers there are only finitely many indecomposables, and in these cases classify them. More generally, one may ask if the possible dimensions of indecomposables can be determined.

The first question was answered by Gabriel in 1972.

THEOREM 0.1 (Gabriel). *A connected quiver admits only finitely many indecomposables if and only if it is an oriented Dynkin graph. In this case the indecomposables are in bijection with the set of positive roots of the simple Lie algebra with the same Dynkin graph. This bijection is given via the dimension of an indecomposable.*

The Dynkin graphs are



where the suffix gives the number of vertices.

We have already considered two examples of type $\mathbb{A}$, namely the one-subspace quiver $\cdot \to \cdot$ of type $\mathbb{A}_2$ and the two-subspace quiver $\cdot \to \cdot \leftarrow \cdot$ of type $\mathbb{A}_3$. If we label the vertices in the Dynkin graph of type $\mathbb{A}$ from left to right by the numbers $1, \ldots, n$, then the positive roots are given by the closed intervals $[i, j]$ for $i \leq j$. Thus there are $\frac{1}{2}n(n+1)$ such positive roots. We see that the positive roots and the dimensions of the indecomposable representations we described above coincide in our two examples, thus verifying Gabriel's Theorem in these two cases.

An answer to the second question was given by Kac in 1982.

THEOREM 0.2 (Kac). *Given an arbitrary (connected) quiver, the set of dimensions of the indecomposables coincides with the set of positive roots of the associated (indecomposable) Kac-Moody Lie algbera (or generalised Kac-Moody Lie algebra if the quiver contains vertex loops).*

This raises the question of how deep the connection between quiver representations and Kac-Moody Lie algebras goes: can we explain Kac's Theorem?

The answer was provided by Ringel in 1990 in the case of Dynkin quivers, and by Green in 1995 for a general quiver. Ringel described how to construct an associative algebra from the category of representations of a given quiver over a fixed finite field k. The structure constants of this algebra reflect the possible extensions in the category. The subalgebra generated by the simple nilpotent representations, the composition algebra, is then isomorphic to the positive part of the quantised enveloping algebra of the associated Lie algebra (specialised at $v^2 = |k|$).

This result was extended by Sevenhant and Van den Bergh in 2001 to show that the whole Ringel-Hall algebra can be viewed as the positive part of the quantised enveloping algebra of a generalised Kac-Moody Lie algebra (although this Lie algebra now depends on the finite field k). Deng and Xiao then showed in 2003 how this approach could be used to deduce Kac's Theorem, by considering the character of the Ringel-Hall algebra on the one hand and the character of the quantised enveloping algebra on the other.

The aims of these lectures are as follows:

- develop the basic representation theory of quivers;
- introduce the Ringel-Hall algebra and study its basic properties;
- prove Green's Formula, showing that the Ringel-Hall algebra is a self-dual Hopf algebra;
- outline the necessary results from quantum groups necessary to prove the isomorphism of Sevenhant and Van den Bergh;
- obtain a presentation of the Ringel-Hall algebra by generators and relations, and give Deng and Xiao's proof of Kac's Theorem;
- show how to use Green's Formula to prove the existence of Hall polynomials for cyclic quivers, Dynkin quivers and all other tame quivers;
- describe the reflection functors and explain the construction due to Bernstein, Gelfand and Ponomarev (1973) of the indecomposable representations for a Dynkin quiver;
- use this theory to describe Poincaré-Birkhoff-Witt bases for the Ringel-Hall algebra in the Dynkin case;
- explain Lusztig's construction of the canonical basis of the Ringel-Hall algebra in the Dynkin case.

Exercises 1.

(1) Classify the indecomposables for the quivers $\cdot \leftarrow \cdot \rightarrow \cdot$ and $\cdot \rightarrow \cdot \rightarrow \cdot$.
(2) Verify Gabriel's Theorem for the quiver of type $\mathbb{A}_n$ with linear orientation (all arrows go from left to right).
(3) Verify Gabriel's Theorem for the three-subspace quiver (of type $\mathbb{D}_4$). There are twelve positive roots in this case.
(4) We know that the four-subspace problem is not a Dynkin quiver. Therefore there exist infinitly many indecomposable representations. Find infinitely many indecomposables for the dimension $(1,1,1,1,2)$ when k is an infinite field.

CHAPTER 1

Representation Theory of Quivers

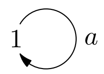
1. Quivers and Representations

A quiver is a finite directed graph, in which we allow multiple edges and vertex loops. More precisely, it is a quadruple $Q = (Q_0, Q_1, t, h)$ consisting of finite sets Q_0 and Q_1 and two maps $t, h\colon Q_1 \to Q_0$. The elements of Q_0 are called the vertices of Q and those of Q_1 the arrows. We draw an arrow $a\colon t(a) \to h(a)$ for each $a \in Q_1$. Examples include

(1) $Q_0 = \{1, 2\}$, $Q_1 = \{a\}$ with $t(a) = 1$ and $h(a) = 2$. Then Q is the one-subspace quiver

$$1 \xrightarrow{a} 2$$

(2) $Q_0 = \{1\}$, $Q_1 = \{a\}$ with $t(a) = h(a) = 1$. Then Q is the Jordan quiver



(3) $Q_0 = \{1, 2, 3\}$, $Q_0 = \{a, b\}$ with $t(a) = 1$, $t(b) = 3$ and $h(a) = h(b) = 2$. Then Q is the two-subspace quiver

$$1 \xrightarrow{a} 2 \xleftarrow{b} 3$$

A subquiver Q' of a quiver Q is given by a pair (Q'_0, Q'_1) such that $t(a), h(a) \in Q'_0$ for each $a \in Q'_1$. A subquiver Q' is called full if Q'_1 contains all arrows $a \in Q_1$ such that $t(a), h(a) \in Q'_0$. The opposite quiver Q^{op} has the same sets Q_0 and Q_1 but with $t^{\mathrm{op}} = h$ and $h^{\mathrm{op}} = t$. The underlying graph of Q is given by forgetting the orientation of the arrows; that is, by repacing each arrow by an edge.

We will always assume that Q is connected; that is, it is not the disjoint union of two non-empty subquivers.

Let Q be a quiver and k a field. A representation of Q is a collection $X = \big(\{X_i\}_{i \in Q_0}, \{X_a\}_{a \in Q_1}\big)$ consisting of a vector space X_i for each vertex i and a linear map $X_a\colon X_{t(a)} \to X_{h(a)}$ for each arrow a. A morphism of representations $\theta\colon X \to Y$ is a collection $\theta = \big(\{\theta_i\}_{i \in Q_0}\big)$ of linear maps $\theta_i\colon X_i \to Y_i$ for each vertex i such that $Y_a \theta_{t(a)} = \theta_{h(a)} X_a$ for each arrow a. In other words, for each arrow a we have a commutative diagram

$$\begin{array}{ccc} X_{t(a)} & \xrightarrow{X_a} & X_{h(a)} \\ \big\downarrow{\scriptstyle \theta_{t(a)}} & & \big\downarrow{\scriptstyle \theta_{h(a)}} \\ Y_{t(a)} & \xrightarrow{Y_a} & Y_{h(a)} \end{array} \tag{1.1}$$

This defines a category $\mathrm{Rep}_k\, Q$. We remark that θ is an isomorphism if and only if each θ_i is an isomorphism. We denote by $\mathrm{rep}_k\, Q$ the full subcategory with objects the finite dimensional representations; that is, those representations X such that each vector space X_i is finite dimensional.

If X is finite dimensional we define the dimension vector of X as

$$\underline{\dim}\, X := \sum_{i \in Q_0} (\dim X_i) e_i \in \mathbb{Z}Q_0. \tag{1.2}$$

Given $\underline{d} = \sum_i d_i e_i \in \mathbb{Z}Q_0$ we write $\mathrm{supp}(\underline{d}) := \{i \in Q_0 : d_i \neq 0\}$. We call $\underline{d}$ a dimension vector if $d_i \geq 0$ for all i.

By choosing bases for each X_i we can represent each linear map X_a by a matrix. It is clear that if two representations X and Y are isomorphic, then we can choose bases for each X_i and each Y_i such that, for each arrow a, the linear maps X_a and Y_a are represented by the same matrix.

Given a dimension vector $\underline{d}$ we form the representation variety

$$\operatorname{Rep}(\underline{d}) := \bigoplus_{a\in Q_1} \mathbb{M}\big(d_{h(a)} \times d_{t(a)}\big). \tag{1.3}$$

To each point $x \in \operatorname{Rep}(\underline{d})$ we have a representation X such that $X_i = k^{d_i}$ and X_a is the linear map associated to the matrix x_a (with respect to the standard bases). We note that $\underline{\dim}\, X = \underline{d}$. Changing bases yields an action of the group

$$\operatorname{GL}(\underline{d}) := \prod_{i\in Q_0} \operatorname{GL}(d_i) \tag{1.4}$$

on the representation variety $\operatorname{Rep}(\underline{d})$. This action is given explicitly by

$$(g\cdot x)_a = g_{h(a)} x_a g_{t(a)}^{-1}. \tag{1.5}$$

We observe that two points in $\operatorname{Rep}(\underline{d})$ give rise to isomorphic representations if and only if they lie in the same $\operatorname{GL}(\underline{d})$-orbit.

Lemma 1.1. *There is a bijection between the isomorphism classes of representations of dimension vector* $\underline{d}$ *and* $\operatorname{GL}(\underline{d})$*-orbits on* $\operatorname{Rep}(\underline{d})$.

Given two dimension vectors $\underline{d}$ and $\underline{e}$ we define

$$\operatorname{Hom}(\underline{d},\underline{e}) := \bigoplus_i \mathbb{M}(e_i \times d_i). \tag{1.6}$$

Given points $x \in \operatorname{Rep}(\underline{d})$ and $y \in \operatorname{Rep}(\underline{e})$ we write

$$\operatorname{Hom}(x,y) := \{\theta \in \operatorname{Hom}(\underline{d},\underline{e}) : y_a\theta_{t(a)} = \theta_{h(a)}x_a \text{ for all } a \in Q_1\}. \tag{1.7}$$

If X and Y are the corresponding representations, then it is clear that

$$\operatorname{Hom}(X,Y) \cong \operatorname{Hom}(x,y), \quad \operatorname{Aut}(X) \cong \operatorname{Hom}(x,x) \cap \operatorname{GL}(\underline{d}) = \operatorname{Stab}_{\operatorname{GL}(\underline{d})}(x). \tag{1.8}$$

2. Path Algebras

A path of length $n \geq 1$ in Q is a sequence of arrows $p = a_1 \cdots a_n$ such that $h(a_r) = t(a_{r+1})$ for each $1 \leq r < n$. We write $t(p) = t(a_1)$ and $h(p) = h(a_n)$. Pictorially, if $i_r = t(a_r) = h(a_{r+1})$, then

$$p\colon i_1 \xrightarrow{a_1} i_2 \xrightarrow{a_2} i_3 - - - i_{n-1} \xrightarrow{a_n} i_n \tag{2.1}$$

Clearly the paths of length 1 are precisely the arrows of Q. For each vertex i there is the trivial path ε_i of length 0 whose head and tail are vertex i.

The path algebra kQ has basis the the set of paths and where $p \cdot q$ is the path given by the concatenation of the sequences of arrows if $h(p) = t(q)$, and is zero otherwise. In particular, the ε_i are pairwise orthogonal idempotents; that is, $\varepsilon_i\varepsilon_j = \delta_{ij}\varepsilon_i$. It follows that the path algebra is an associative unital algebra, with unit $1 = \sum_i \varepsilon_i$.

Let Q_r denote the set of paths of length r. This extends the notation for the vertices Q_0 and the arrows Q_1. We have

$$kQ = \bigoplus_{r\geq 0} kQ_r, \tag{2.2}$$

where kQ_r is the vector space with basis the elements of Q_r. Also, by construction,

$$kQ_r \cdot kQ_s = kQ_{r+s}. \tag{2.3}$$

Thus kQ is an $\mathbb{N}_0$-graded algebra.

Examples.

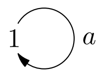
(1) Let Q be the one-subspace quiver

$$1 \xrightarrow{a} 2$$

Then $kQ \cong \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subset \mathbb{M}(2 \times 2)$, where

$$\varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(2) Let Q be the Jordan quiver



Then $kQ \cong k[T]$, where $\varepsilon_1 = 1$ and $a = T$.

LEMMA 1.2. (1) *$\varepsilon_i kQ$ has basis those paths p with $t(p) = i$. Dually, $kQ\varepsilon_i$ has basis those paths p with $h(p) = i$.*
(2) *$\varepsilon_i kQ \varepsilon_j$ has basis those paths p with $t(p) = i$ and $h(p) = j$.*
(3) *$kQ\varepsilon_i kQ$ has basis those paths passing through i.*

LEMMA 1.3. *The ε_i form a complete set of pairwise inequivalent primitive orthogonal idempotents in kQ. In particular, $kQ_0 = \prod_i k\varepsilon_i$ is a semisimple algebra and the modules $P_i = \varepsilon_i kQ$ are pairwise non-isomorphic indecomposable projective modules.*

PROOF. We have already remarked that the ε_i form a complete set of orthogonal idempotents. Let x be any idempotent. We can write $x = x_0 + x'$ where x_0 is the homogeneous part of degree 0. Then $x_0 = 0$ implies $x = x' = 0$, by degree considerations.

To show that the ε_i are primitive, suppose that $\varepsilon_i = x + y$ is a sum of orthogonal idempotents. Write $x = x_0 + x'$ and $y = y_0 + y'$ as above. Then $\varepsilon_i = x_0 + y_0$ is sum of orthogonal idempotents of degree 0, hence $x_0 = 0$ or $y_0 = 0$ and so $x = 0$ or $y = 0$.

To show that they are inequivalent, suppose we can write $\varepsilon_i = xy$ and $\varepsilon_j = yx$. Writing $x = x_0 + x'$ and $y = y_0 + y'$ as before, we see that $\varepsilon_i = x_0 y_0$ and $\varepsilon_j = y_0 x_0$. It follows that $i = j$. □

PROPOSITION 1.4. *The category $\mathrm{Rep}_k Q$ is equivalent to the category $\mathrm{Mod}\, kQ$ of all right kQ-modules. Similarly, $\mathrm{rep}_k Q$ is equivalent to $\mathrm{mod}\, kQ$. In particular, both $\mathrm{Rep}_k Q$ and $\mathrm{rep}_k Q$ are abelian.*

PROOF. Given a representation X of Q, define a kQ-module FX as follows. As a vector space set $FX := \bigoplus_i X_i$. Let $\pi_i \colon FX \to X_i$ and $\iota_i \colon X_i \to FX$ be the canonical projection and inclusion maps with respect to the direct sum decomposition of FX. Now define $x \cdot \varepsilon_i := \iota_i \pi_i(x)$ and for $p = a_1 \cdots a_n \in Q_n$

$$x \cdot p := \iota_{h(p)} X_{a_n} \cdots X_{a_1} \pi_{t(p)}(x).$$

Extending linearly endows FX with the structure of a right kQ-module. If $\theta\colon X \to Y$ is a morphism of representations we define $F\theta := \bigoplus_i \theta_i$ to be the direct sum of linear maps; that is,

$$F\theta(x_i)_{i\in Q_0} = (\theta_i(x_i))_{i\in Q_0}.$$

Thus F determines a functor $\operatorname{Rep}_k Q \to \operatorname{Mod} kQ$.

Conversely we define a functor $G\colon \operatorname{Mod} kQ \to \operatorname{Rep}_k Q$ as follows. For a module M we define GM via

$$(GM)_i := M\cdot\varepsilon_i \quad\text{and}\quad (GM)_a\colon M\cdot\varepsilon_{t(a)} \to M\cdot\varepsilon_{h(a)}, \quad m\mapsto m\cdot a.$$

If $\phi\colon M \to N$ is a morphism, then $G\phi\colon GM \to GN$ is defined via restriction; that is, for $m\in M\cdot\varepsilon_i$ we have $\phi(m) = \phi(m\cdot\varepsilon_i) = \phi(m)\cdot\varepsilon_i \in N\cdot\varepsilon_i$.

Now, $GF = \mathrm{id}$ on $\operatorname{Rep}_k Q$ and $FG \cong \mathrm{id}$ on $\operatorname{Mod} kQ$. The difference lies in the fact that $FG(M) = \bigoplus_i M\cdot\varepsilon_i$ is an outer direct sum, so we have natural isomorphism $M \to FG(M)$, but not equality.

Clearly F and G preserve finite dimensionality, so induce equivalences between $\operatorname{rep}_k Q$ and $\operatorname{mod} kQ$. □

It follows that we can talk about subrepresentations and direct sums of representations, hence also about indecomposable representations. Furthermore we have the notions of kernels, images and cokernels of morphisms and of short exact sequences in the category $\operatorname{Rep}_k Q$, and can apply the techniques of homological algebra.

As an example, we have for each vertex i a simple representation S_i. This has vector space k at vertex i, all other vector spaces 0 and all linear maps 0. These are not the only simple representations, however. Consider the Jordan quiver. Then, for each $\lambda\in k$ we have a simple representation R_λ with vector space k and linear map given by multiplication by λ. The simple S constructed above coincides with the representation R_0.

A more interesting example is to consider the indecomposable projective module $P_i = \varepsilon_i kQ$. This has as basis the set of paths with tail i. What does the corresponding representation look like?

We consider this for the quiver $Q\colon 1 \xrightarrow{a} 2$. Then $kQ = \begin{pmatrix} k & k \\ 0 & k\end{pmatrix}$, so P_2 has basis ε_2 and P_1 has basis ε_1, a. Consider the functor G from the proof of Proposition 1.4. Then

$$P_2\cdot\varepsilon_1 = 0, \quad P_2\cdot\varepsilon_2 = k\varepsilon_2,$$

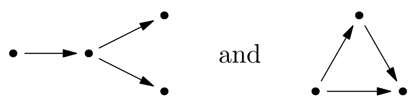
so that P_2 is the representation $0\to k$ with dimension vector $\underline{\dim}\, P_2 = e_2$. Similarly,

$$P_1\cdot\varepsilon_1 = k\varepsilon_1, \quad P_1\cdot\varepsilon_2 = ka,$$

and multiplication by a sends basis vector ε_1 to a. Thus P_1 is the representation $k \xrightarrow{1} k$ with dimension vector $\underline{\dim}\, P_1 = e_1 + e_2$.

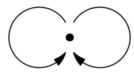
Exercises 2.

(1) Calculate the indecomposable projectives for the quivers



(2) Recall that Q is connected. Show that the centre of kQ equals $k[T]$ if Q is an oriented cycle, and k otherwise.

(3) Prove that kQ is right noetherian if and only if, for all oriented cycles C in Q and all vertices $i \in C$, there is precisely one arrow with tail i. Dually kQ is left noetherian if and only if, for all oriented cycles C in Q and all vertices $i \in C$, there is precisely one arrow with head i. Classify those quivers which are both left and right noetherian.
(4) Show that $\varepsilon_i kQ \varepsilon_i$ is isomorphic to the path algebra of a quiver which has a unique vertex and one loop for each primitive cycle of Q starting at vertex i. (A primitive cycle is a cycle which is not the product of two smaller cycles.) Let ε be any idempotent. Show that $\varepsilon kQ \varepsilon$ is again isomorphic to the path algebra of a (possibly infinite) quiver. Give an example of a finite quiver Q and an idempotent ε such that $\varepsilon kQ \varepsilon$ is the path algebra of an infinite quiver.
(5) Consider the double loop quiver Q



Show that kQ has simple modules of each dimension $n \geq 1$. Find an infinite dimensional simple module.

3. Krull-Remak-Schmidt Theorem

In this section we work over an arbitrary k-algebra A.

Lemma 1.5 (Fitting). *For $M \in \operatorname{mod} A$ we have that M is indecomposable if and only* $\operatorname{End}(M)$ *is local; that is, every endomorphism is either an automorphism or nilpotent.*

Proof. Let $\dim M = n$ be a finite dimensional module and $f \in \operatorname{End}(M)$ an A-endomorphism of M. We can decompose M into its generalised eigenspaces with respect to f, so $M = M_1 \oplus \cdots \oplus M_r$ where $M_i = \operatorname{Ker}(p_i(f))^n$ for some monic irreducible polynomial $p_i(t) \in k[t]$.

Since f is an A-endomorphism of M, the same is true of each polynomial in f. In particular, each $(p_i(f))^n$ is an A-endomorphism of M, so that the kernel is a A-submodule. Thus the decomposition into generalised eigenspaces is also a direct sum decomposition in $\operatorname{mod} A$.

Now, if M is indecomposable, then each endomorphism must have characteristic polynomial of power of a unique irreducible polynomial. An endomorphism is thus nilpotent if its characteristic polynomial equals t^n, and is an automorphism otherwise.

Conversely, if $M = M_1 \oplus M_2$ is decomposable, then the projection maps give rise to two non-zero orthogonal idempotents in $\operatorname{End}(M)$, and these both have eigenvalues 0 and 1. $\square$

Corollary 1.6. *Let M be indecomposable, $f, g \in \operatorname{End}(M)$ nilpotent and $\theta \in \operatorname{End}(M)$ an automorphism. Then $f + g$ is nilpotent and $f + \theta$ is an automorphism.*

Proof. Let f and g be nilpotent, say $f^n = 0 = g^n$, and let θ be an automorphism. Clearly

$$(1 - f)(1 + f + f^2 + \cdots + f^{n-1}) = 1,$$

so $1 - f$ is an automorphism of M. Also $-\theta^{-1} f$ is nilpotent, since it has eigenvalue 0. Hence $\theta + f = \theta(1 + \theta^{-1} f)$ is an automorphism.

Suppose that $f+g=\theta$ is an automorphism. Then $f=\theta-g$, and $\theta-g$ is an automorphism whilst f is nilpotent, a contradiction. Hence $f+g$ is nilpotent. □

For modules M and X with X indecomposable we define

$$\operatorname{Rad}\operatorname{Hom}(X,M) := \{f \in \operatorname{Hom}(X,M) : gf \text{ nilpotent } \forall g \in \operatorname{Hom}(M,X)\}. \quad (3.1)$$

Lemma 1.7. *Let M, N, X and Y be modules with X and Y indecomposable. Then*

(1) $\operatorname{Rad}\operatorname{Hom}(X,M)$ *is a subspace of* $\operatorname{Hom}(X,M)$.
(2) $\operatorname{Rad}\operatorname{Hom}(X,M\oplus N) = \operatorname{Rad}\operatorname{Hom}(X,M) \oplus \operatorname{Rad}\operatorname{Hom}(X,N)$.
(3) $\operatorname{Rad}\operatorname{Hom}(X,Y) = \operatorname{Hom}(X,Y)$ *if* $X \not\cong Y$.

Proof. Let $f, f' \in \operatorname{Rad}\operatorname{Hom}(X,M)$ and $g \in \operatorname{Hom}(M,X)$. Then $g(f+f') = gf + gf' \in \operatorname{End}(X)$ is nilpotent since both gf and gf' are nilpotent.

We have the natural isomorphisms $\operatorname{Hom}(X,M\oplus N) \cong \operatorname{Hom}(X,M)\oplus\operatorname{Hom}(X,N)$ and $\operatorname{Hom}(M\oplus N,X) \cong \operatorname{Hom}(M,X)\oplus\operatorname{Hom}(N,X)$. Let $f=(f_1,f_2) \in \operatorname{Hom}(X,M\oplus N)$ and $g=(g_1,g_2) \in \operatorname{Hom}(M,X)\oplus\operatorname{Hom}(N,X)$. Then $gf = g_1f_1 + g_2f_2$. If f is a radical morphism, then considering those g with either $g_2=0$ or $g_1=0$ we deduce that both f_1 and f_2 are radical. Conversely, if both f_1 and f_2 are radical, then g_1f_1 and g_2f_2 are nilpotent, so their sum gf is nilpotent, hence f is a radical morphism.

Finally, let Y be indecomposable and consider morphisms $f\colon X \to Y$ and $g\colon Y \to X$. Then $gf \in \operatorname{End}(X)$ is either nilpotent or an automorphism, and similarly for $fg \in \operatorname{End}(Y)$. Now, if fg is nilpotent, say $(fg)^n = 0$, then $(gf)^{n+1} = g(fg)^n f = 0$ is nilpotent, and *vice versa*. Thus fg and gf are either both nilpotent or both automorphisms. If fg and gf are both automorphisms, then f and g must both be isomorphisms, hence $Y \cong X$. □

Theorem 1.8 (Krull-Remak-Schmidt). *Let M be a finite dimensional module. Then we can write $M \cong X_1^{a_1} \oplus \cdots \oplus X_r^{a_r}$ with the X_i pairwise non-isomorphic indecomposable modules and each $a_i \geq 1$. If $M \cong Y_1^{b_1} \oplus \cdots Y_s^{b_s}$ is another such decomposition, then $r=s$ and, after reordering, $X_i \cong Y_i$ and $a_i = b_i$.*

Proof. Induction on dimension shows that every $M \in \operatorname{mod} kQ$ decomposes into a finite direct sum of indecomposable modules. Suppose that $M \cong X_1^{a_1} \oplus \cdots \oplus X_r^{a_r}$ is such a direct sum decomposition with the X_i pairwise non-isomorphic indecomposable modules and each $a_i \geq 1$.

Let Y be indecomposable and consider $\frac{\dim \operatorname{Hom}(Y,M) - \dim \operatorname{Rad}\operatorname{Hom}(Y,M)}{\dim \operatorname{End}(Y) - \dim \operatorname{Rad}\operatorname{End}(Y)}$. We see by the previous lemma that this number equals a_i if $Y \cong X_i$ (there is at most one such i) and 0 otherwise. In particular, this number is independent of the decomposition. □

Exercises 3.

(1) Consider the four subspace quiver, with central vertex 0 and other vertices $1,2,3,4$. Define a representation of dimension vector $2re_0 + re_1 + re_2 + re_3 + (r+1)e_4$, where the matrices are given by

$$\begin{pmatrix} 1_r \\ 0_r \end{pmatrix}, \quad \begin{pmatrix} 0_r \\ 1_r \end{pmatrix}, \quad \begin{pmatrix} 1_r \\ 1_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_r & 0' \\ 0' & 1_r \end{pmatrix}.$$

Here we have written 0_r and 1_r respectively for the zero and the identity $r \times r$-matrices, and $0'$ for the zero $r \times 1$-matrix. Show that the endomorphism algebra of this representation is just k.

(2) Let A be a finitely generated k-algebra, say with generators $g_1, \dots, g_r$. Denote by $\iota_i \colon A \to A^{2r}$ the canonical map onto the i-th component. Consider the five subspace quiver and define a representation as follows. Let V be the representation of the four subspace quiver constructed above, such that $\dim V_0 = 2r$. Let $A \otimes_k V$ be the representation given by taking the tensor product with A at each vertex. Define the fifth subspace via $A \mapsto A^{2r} = (A \otimes_k V)_0$, $a \mapsto \sum_{i=1}^r \big(\iota_i(a) + \iota_{i+r}(g_i a)\big)$. Show that the endomorphism algebra of this representation is isomorphic to the set of all k-automorphisms of A which commute with each of the endomorphisms $a \mapsto g_i a$. Hence show that this algebra is isomorphic to A^{op}. (This construction is due to S. Brenner.)

(3) Show that all finitely generated k-algebras arise as the endomorphism algebra of some module over the three Kronecker quiver.

4. Heredity and Tensor Algebras

We have already observed that $kQ = \bigoplus_{r\geq 0} kQ_r$ is a graded algebra such that $kQ_0 \cong \prod_i k\varepsilon_i$ is semisimple and $kQ_r \cdot kQ_s = kQ_{r+s}$. In this section we show that kQ is actually the tensor algebra of the bimodule kQ_1 over kQ_0. Moreover, we show that all such tensor algebras Λ are hereditary algebras.

A ring is called hereditary if each module in $\operatorname{Mod}\Lambda$ has a projective resolution of length at most 1, i.e. $\operatorname{gl\,dim}\Lambda \leq 1$. This is equivalent to saying that every submodule of a projective module is again projective, hence the term hereditary.

Let Λ_0 be a semisimple ring and Λ_1 a finite length Λ_0-bimodule. The tensor ring $T(\Lambda_0, \Lambda_1)$ is the $\mathbb{N}$-graded Λ_0-module

$$\Lambda := \bigoplus_{r\geq 0} \Lambda_r, \quad \text{where} \quad \Lambda_r := \Lambda_1 \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} \Lambda_1 \ (r \text{ times}) \tag{4.1}$$

and with multiplication given via the natural isomorphism $\Lambda_r \otimes_{\Lambda_0} \Lambda_s \cong \Lambda_{r+s}$. If $\lambda \in \Lambda_r$ is homogeneous, we write $|\lambda| = r$ for its degree.

In the case of a path algebra kQ, we see that each kQ_r is a kQ_0-bimodule and that as bimodules, $kQ_r \cong kQ_1 \otimes_{kQ_0} \cdots \otimes_{kQ_0} kQ_1$ (r times). Under this identification, the multiplication in the path algebra is precisely the concatenation of tensors, thus the path algebra is an example of a tensor ring.

Let Λ be a tensor ring as above. The graded radical of Λ is the ideal $\Lambda_+ := \bigoplus_{r\geq 1} \Lambda_r$. Note that $\Lambda_+ \cong \Lambda_1 \otimes_{\Lambda_0} \Lambda$ as a right Λ-module.

Theorem 1.9. *Let Λ be a tensor ring and $M \in \operatorname{Mod}\Lambda$.*

(1) *There is a projective resolution of M*

$$0 \to M \otimes_{\Lambda_0} \Lambda_+ \xrightarrow{\delta_M} M \otimes_{\Lambda_0} \Lambda \xrightarrow{\epsilon_M} M \to 0$$

where, for $m \in M$, $\lambda \in \Lambda$ and $\mu \in \Lambda_1$,

$$\epsilon_M(m \otimes \lambda) := m \cdot \lambda,$$
$$\delta_M(m \otimes (\mu \otimes \lambda)) := m \otimes (\mu \otimes \lambda) - m \cdot \mu \otimes \lambda.$$

(2) *There is an injective resolution of M*

$$0 \to M \xrightarrow{\epsilon^M} \operatorname{Hom}_{\Lambda_0}(\Lambda, M) \xrightarrow{\delta^M} \operatorname{Hom}_{\Lambda_0}(\Lambda_+, M) \to 0$$

where, for $m \in M$, $\lambda \in \Lambda$ and $\mu \in \Lambda_1$,

$$\epsilon^M(m)(\lambda) := m \cdot \lambda,$$
$$\delta^M(f)(\lambda \otimes \mu) := f(\lambda \otimes \mu) - f(\lambda) \cdot \mu.$$

In particular, Λ is hereditary.

PROOF. We prove only the first statement.

It is clear that ϵ_M is an epimorphism and that $\epsilon_M \delta_M = 0$. To see that δ_M is a monomorphism, we decompose $M \otimes \Lambda = \bigoplus_{r \geq 0} M \otimes \Lambda_r$, and similarly for $M \otimes \Lambda_+$. Then δ_M restricts to maps $M \otimes \Lambda_r \to (M \otimes \Lambda_r) \oplus (M \otimes \Lambda_{r-1})$ for each $r \geq 1$, and moreover acts as the identity on the first component. In particular, if $\sum_{r=1}^{t} x_r \in \operatorname{Ker}(\delta_M)$ with $x_r \in M \otimes \Lambda_r$, then $x_t = 0$. Thus the kernel of δ_M is trivial. Similar considerations show that $M \otimes \Lambda = \big(M \otimes \Lambda_0\big) \oplus \operatorname{Im}(\delta_M)$, whence $\operatorname{Ker}(\epsilon_M) = \operatorname{Im}(\delta_M)$.

It remains to show that this is a projective resolution. Since Λ_0 is semisimple, each Λ_0-module is projective. Thus $M \otimes_{\Lambda_0} \Lambda$ and $M \otimes_{\Lambda_0} \Lambda_+ \cong \big(M \otimes_{\Lambda_0} \Lambda_1\big) \otimes_{\Lambda_0} \Lambda$ are both projective Λ-modules. □

THEOREM 1.10 (Wedderburn). *Let Γ be a finite dimensional algebra over a perfect field k. Then there exists a tensor algebra Λ and an ideal I such that $\Gamma \cong \Lambda/I$ and $\Lambda_+^r \subset I \subset \Lambda_+^2$ for some r.*

PROOF. Let J denote the Jacobson radical of Γ, so that Γ/J is a semisimple k-algebra. Since k is a perfect field, this is a separable algebra, hence there exists a subalgebra $\Gamma_0 \subset \Gamma$ with $\Gamma = \Gamma_0 \oplus J$ (as Γ_0-bimodules).

Moreover, we can consider the natural epimorphism $J \to J/J^2$. Again, this is split as Γ_0-bimodules (since $\Gamma_0^e = \Gamma_0 \otimes_k \Gamma_0^{\mathrm{op}}$ is semisimple). Thus we can write $J = \Gamma_1 \oplus J^2$ as Γ_0-bimodules.

Finally we see that Γ is generated as a k-algebra by Γ_0 and Γ_1, since J is nilpotent.

Define $\Lambda := T(\Gamma_0, \Gamma_1)$. There is a surjective algebra homomorphism $\Lambda \to \Gamma$, using the splittings found above. Denote the kernel by I. Then clearly $I \subset \Lambda_+^2$ and $\Lambda_+/I = J$. This is nilpotent, so $\Lambda_+^r \subset I$ for some r. □

THEOREM 1.11. *Let Λ be a tensor k-algebra and $I \subset \Lambda_+^2$ an ideal such that Λ/I is hereditary. Assume further that either I is graded, or else $\Lambda_+^r \subset I$ for some r. Then $I = 0$.*

PROOF. Let $\Gamma = \Lambda/I$ and set $\Gamma_+ = \Lambda_+/I$. If I is graded, then Γ is again graded and Γ_+ denotes the graded radical. On the other hand, if $\Lambda_+^r \subset I$, then Γ is finite dimensional and Γ_+ equals the Jacobson radical.

Consider the short exact sequence of Γ-modules

$$0 \to \Gamma_+ \to \Gamma \to \Gamma/\Gamma_+ \to 0.$$

Since Γ is hereditary, Γ_+ must be projective, hence the following short exact sequence of Γ-modules is split:

$$0 \to I/\Lambda_+ I \to \Lambda_+/\Lambda_+ I \to \Lambda_+/I \to 0.$$

It follows that there exists a right Λ-submodule $M \subset \Lambda_+$ such that $\Lambda_+ = M + I$ and $M \cap I = \Lambda_+ I$. Since $I \subset \Lambda_+^2$, we have $\Lambda_+ = M + \Lambda_+^2$. Then $\Lambda_+^2 = M\Lambda_+ + \Lambda_+^3 \subset M + \Lambda_+^3$, whence $\Lambda_+ = M + \Lambda_+^3$. By induction, $\Lambda_+ = M + \Lambda_+^n$ for any $n \geq 2$.

If I is graded, then we may further take M to be a graded Λ-module, and from the above, $\Lambda_n \subset M$ for all $n \geq 1$. Thus $M = \Lambda_+$. If $\Lambda_+^r \subset I$, then $\Lambda_+^{r+1} \subset \Lambda_+ I \subset M$, so that $\Lambda_+ \subset M + \Lambda_+^{r+1} \subset M$ and again $M = \Lambda_+$.

In both cases we have that $I = M \cap I = \Lambda_+ I$. Since $I \subset \Lambda_+^2$, we see that $I \subset \Lambda_+^n$ for all $n \geq 2$. Hence $I = 0$. □

Up to Morita equivalence, we may always assume that Λ_0 is basic semisimple.

Note that not all finite dimensional hereditary k-algebras are tensor algebras. Let A be a basic finite dimensional hereditary k-algebra with Jacobson radical J. Let $1 = \varepsilon_1 + \cdots + \varepsilon_n$ be a decomposition into pairwise orthogonal primitive idempotents. Then one can order these idempotents such that $\varepsilon_i J \varepsilon_j = 0$ for $i \geq j$. In particular, $J = \sum_{i<j} \varepsilon_i J \varepsilon_j$ and $B := \sum_i \varepsilon_i A \varepsilon_i$ is a semisimple subalgebra satisfying $A = B \oplus J$. The question is thus: does $J \to J/J^2$ split as B-bimodules? Generally the answer is no.

Such an example was constructed by Dlab and Ringel. Let K be a field, σ an automorphism of K and δ a $(\sigma, 1)$-derivation; that is, δ is additive and $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$. Let M be a K-bimodule, isomorphic to $K \oplus K$ as a left module, and with the right k-action given via $(a, b) \cdot c := (ac + b\delta(c), b\sigma(c))$. Let A be the $n \times n$-upper triangular matrix ring given by taking a copy of K in each position except $(1, n)$, where we take a copy of the bimodule M. This ring is hereditary and semiprimary, but is a tensor ring if and only if δ is an inner derivation. Moreover, if δ is identically zero on $k \subset K^\sigma$, then A is even a k-algebra.

In fact, for this algebra, the representation type of A cannot be determined by A/J^2.

The result is true, however, if A is representation finite (Ringel). It also holds if the quiver of A does not contain a subquiver of the form $\tilde{\mathbb{A}}_n$ with all but one arrow pointing clockwise (Dlab-Ringel).

Exercises 4.

(1) Let $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ be an $\mathbb{N}_0$-graded algebra. A $\mathbb{N}_0$-graded module M is a Λ-module with a vector space decomposition $M = \bigoplus_{n \geq 0} M_n$ such that $M_r \cdot \Lambda_s \subset M_{r+s}$. Describe the graded simple modules for a path algebra kQ. Prove that each finite dimensional graded module has a filtration by such simples. Show that the converse does not hold: there exist modules which admit a filtration by the graded simple modules but which are not themselves gradable.

(2) Let $\Lambda = T(\Lambda_0, \Lambda_1)$ be the tensor algebra of a bimodule Λ_1 over a semisimple algebra Λ_0. Prove that $\operatorname{Mod} \Lambda$ is isomorphic to the category whose objects consist of pairs (M, μ_M) with M a Λ_0-module and $\mu_M \colon M \otimes_{\Lambda_0} \Lambda_1 \to M$ a Λ_0-homomorphism, and whose morphisms $f \colon (M, \mu_M) \to (N, \mu_N)$ are given by those Λ_0-morphisms $f \colon M \to N$ such that $\mu_N(f \otimes 1) = f\mu_M$.

(3) For $\Lambda = kQ$ a path algebra, translate the first part of Theorem 1.9 into the language of representations.

(4) Prove the second part of Theorem 1.9 concerning injective resolutions.

5. k-Species and the Euler Form

Let k be a field. By definition, a k-species is a tensor algebra $\Lambda = T(\Lambda_0, \Lambda_1)$ such that Λ_0 is a basic semisimple finite dimensional k-algebra and Λ_1 is a finite dimensional Λ_0-bimodule on which k acts centrally. This is equivalent to saying that Λ_1 is a $\Lambda_0^e = \Lambda_0 \otimes_k \Lambda_0^{\mathrm{op}}$-module, and implies that Λ is a k-algebra.

Recall that the category of Λ-modules is isomorphic to the category of pairs (M, μ_M) where M is a Λ_0-module and $\mu_M \colon M \otimes_{\Lambda_0} \Lambda_1 \to M$ is a Λ_0-homomorphism.

We record the fact that the path algebras introduced earlier are precisely those k-species with $\Lambda_0 = \prod_i k$ is a product of copies of the base field.

Lemma 1.12 (Ringel). *Let (M, μ_M) and (N, μ_N) be two Λ-modules. Consider the map*

$$\gamma \colon \mathrm{Hom}_{\Lambda_0}(M, N) \to \mathrm{Hom}_{\Lambda_0}(M \otimes_{\Lambda_0} \Lambda_1, N), \quad f \mapsto \mu_N(f \otimes 1) - f\mu_M.$$

Then $\mathrm{Ker}(\gamma) \cong \mathrm{Hom}_\Lambda(M, N)$ *and* $\mathrm{Coker}(\gamma) \cong \mathrm{Ext}^1_\Lambda(M, N)$.

Proof. The kernel of γ is given by those f such that $\mu_N(f \otimes 1) = f\mu_M$, which by definition is the set of Λ-homomorphisms $M \to N$.

Given a map $g \colon M \otimes_{\Lambda_0} \Lambda_1 \to N$ we define an extension $\eta_g \in \mathrm{Ext}^1(M, N)$ as follows. We set $E := M \oplus N$ as a Λ_0-module and define $\mu_g \colon E \otimes_{\Lambda_0} \Lambda_1 \to E$ via

$$\mu_g := \begin{pmatrix} \mu_M & 0 \\ g & \mu_N \end{pmatrix} : (M \otimes_{\Lambda_0} \Lambda_1) \oplus (N \otimes_{\Lambda_0} \Lambda_1) \otimes_{\Lambda_0} \Lambda_1 \to M \oplus N.$$

Then $E_g := (E, \mu_g)$ defines a Λ-module, and the natural Λ_0-morphisms

$$\iota = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : N \to E \quad \text{and} \quad \pi = \begin{pmatrix} 0 & 1 \end{pmatrix} : E \to M$$

show that E_g is an extension of M by N. We define $\eta_g \in \mathrm{Ext}^1_\Lambda(M, N)$ to be the class of this extension. Conversely, any short exact sequence $0 \to N \to E \to M \to 0$ of Λ-modules is split over Λ_0, so every extension class is of the form η_g for some g. Thus there is a surjective map $\mathrm{Hom}_{\Lambda_0}(M \otimes_{\Lambda_0} \Lambda_1, N) \to \mathrm{Ext}^1_\Lambda(M, N)$.

It remains to show that this is the cokernel of γ, so suppose $g \colon M \otimes_{\Lambda_0} \Lambda_1 \to N$ is such that $\eta_g = 0$. Then there exists a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\iota} & E_g & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{\iota} & M \oplus N & \xrightarrow{\pi} & M & \longrightarrow & 0 \end{array}$$

where $M \oplus N$ has the natural Λ-module structure given via $\mu_M \oplus \mu_N$. Writing

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} : M \oplus N \to M \oplus N$$

as a matrix of Λ_0-homomorphisms, we see that $h_{11} = 1$, $h_{22} = 1$ and $h_{12} = 0$, by the commutativity of the two squares. Moreover, since h corresponds to a Λ-homomorphism $E_g \to M \oplus N$, we must have

$$\mu_N(h_{21} \otimes 1) = h_{21}\mu_M + g.$$

Thus $g = \mu_N(h_{21} \otimes 1) - h_{21}\mu_M = \gamma(h_{21})$.

Conversely, if $g = \gamma(h_{21})$, then it is clear from the above constructions that $\eta_g = 0$. □

Since Λ_0 is basic semisimple, we can express it as

$$\Lambda_0 = \prod_{i \in Q_0} k_i, \quad k_i \text{ a skew-field.} \tag{5.1}$$

We denote the unit in k_i as ε_i. We can also decompose Λ_1 as

$$\Lambda_1 = \bigoplus_{i,j\in Q_0} k_{ij}, \quad k_{ij} = \varepsilon_i \Lambda_1 \varepsilon_j \text{ a } k_i\text{-}k_j\text{-bimodule.} \tag{5.2}$$

By definition, k is contained in the centre of each k_i and acts centrally on each k_{ij}. Moreover, each k_i and k_{ij} is finite dimensional over k. We write

$$s_i := \dim_k k_i \quad \text{and} \quad a_{ij} := \dim_k k_{ij}. \tag{5.3}$$

Note that

$$s_i \dim_{k_i} k_{ij} = a_{ij} = s_j \dim_{k_j} k_{ij}.$$

Define a matrix $R = R_\Lambda$ via

$$R := \big(s_i \delta_{ij} - a_{ij}\big)_{i,j\in Q_0}. \tag{5.4}$$

The dimension vector of a Λ-module M is defined as

$$\underline{\dim}\, M := \sum_{i\in Q_0} \big(\dim_{k_i} M\cdot\varepsilon_i\big) e_i \in \mathbb{Z}Q_0. \tag{5.5}$$

COROLLARY 1.13. *For M and N finite dimensional Λ-modules, we have*

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle := (\underline{\dim}\, M) R (\underline{\dim}\, N)^t = \dim \operatorname{Hom}_\Lambda(M,N) - \dim \operatorname{Ext}^1_\Lambda(M,N).$$

We shall also need the symmetrisation of this form, given by

$$(\underline{d},\underline{e}) := \langle \underline{d},\underline{e}\rangle + \langle \underline{e},\underline{d}\rangle = (\underline{d})(R+R^t)(\underline{e})^t. \tag{5.6}$$

As for quivers, there is a natural parameterising space for the modules of dimension vector $\underline{d}$. As a Λ_0-module, this is naturally isomorphic to the k-vector space $M = \coprod_i k_i^{d_i}$. Using the decomposition of Λ_1 given above, a module structure on this space is completely determined by the maps $\mu_{ij}\colon k_i^{d_i} \otimes_{k_i} k_{ij} \to k_j^{d_j}$. We therefore take as parametrising space

$$\operatorname{Rep}(\underline{d}) := \bigoplus_{i,j} \operatorname{Hom}_{k_j}\big(k_{ij}^{d_i}, k_j^{d_j}\big). \tag{5.7}$$

The notion of isomorphism translates into an action of the group

$$\operatorname{GL}(\underline{d}) := \prod_i \operatorname{GL}(d_i, k_i) \tag{5.8}$$

on the space $\operatorname{Rep}(\underline{d})$, where the action is given by conjugation. We have used here the identification $k_{ij}^{d_i} = k_i^{d_i} \otimes_{k_i} k_{ij}$. Just as for quivers, the orbits are in bijection with the isomorphism classes of modules of dimension vector $\underline{d}$, and the stabiliser of a point is isomorphic to the automorphism group of the corresponding module.

Of particular interest for these lectures is the case of a finite field k. We record the following lemma determining the structure of the algebra $k_i \otimes_k k_j$, and hence the possible simple bimodules which can occur as summands of k_{ij}.

LEMMA 1.14. *Let k be a finite field with algebraic closure $\bar{k}$. Let k_i, k_j, K and L be field extensions of k contained in $\bar{k}$ of degrees s_i, s_j, $\gcd(s_i,s_j)$ and $\operatorname{lcm}(s_i,s_j)$ respectively. Then*

$$k_i \otimes_k k_j = \bigoplus_{[\sigma]\in G} L_{[\sigma]} \quad \textit{where} \quad G := \operatorname{Gal}(L,k)/\operatorname{Gal}(L,K) \cong \operatorname{Gal}(K,k).$$

The k_i-k_j-bimodule structure on $L_{[\sigma]}$ is given by $a\cdot\lambda\cdot b := \sigma(a)\lambda b$.

PROOF. We know that $k_i \otimes_k k_j \cong k_i \otimes_K (K \otimes_k K) \otimes_K k_j$ and that $K \otimes_k K \cong \bigoplus_{\tau \in \mathrm{Gal}(K,k)} K_\tau$. Here, K_τ is the K-K-bimodule which, as a set, is given by K, and with action $a \cdot \lambda \cdot b := \tau(a)\lambda b$.

Let $\sigma \in \mathrm{Gal}(L,k)$ be a lift of τ. There is a map $k_i \otimes_K K_\tau \otimes_K k_j \to L_\sigma$ of k_i-k_j-bimodules sending $\lambda \otimes \mu \otimes \nu \mapsto \sigma(\lambda)\mu\nu$. Since this is surjective and the dimensions over k agree, this is an isomorphism. In particular, L_σ, as a $k_i \otimes_k k_j$-module, depends only on the class $[\sigma] \in G$. □

The matrix R_Λ gives us our first connection to Lie algebras.

Consider $R_{\Lambda^{\mathrm{op}}}$, the matrix arising from the opposite algebra. Since this only depends on dimensions, we see that $R_{\Lambda^{\mathrm{op}}} = R_\Lambda^t$, the transpose. Thus $B := R + R^t = \big(2s_i\delta_{ij} - (a_{ij} + a_{ji})\big)$ is a symmetric integer matrix, which depends only on the dimensions $s_i = \dim_k k_i$ and $a_{ij} + a_{ji} = \dim_k(k_{ij} \oplus k_{ji})$. In particular, if $\Lambda = kQ$ is a path algebra, we see that B is "independent of the orientation of Q"; that is, any other orientation of the same underlying graph yields the same matrix B.

Note that the symmetric bilinear form $(-,-)$ defined above is given by the matrix B.

Define now $D := \mathrm{diag}(s_i)$, an invertible, diagonal matrix, and $C := D^{-1}B$. We note that

$$\begin{aligned} c_{ii} &= 2(1 - a_{ii}/s_i) \in 2\mathbb{Z} \text{ and } c_{ii} \leq 2; \\ c_{ij} &\leq 0 \text{ for } i \neq j; \\ s_i c_{ij} &= s_j c_{ji}. \end{aligned} \tag{5.9}$$

Thus C is a finite symmetrisable Borcherds matrix. (See Section 1 for a general definition.) Note that a symmetrisable generalised Cartan matrix is a matrix as above satisfying the extra condition that $c_{ii} = 2$ for all i.

LEMMA 1.15. *Every such finite symmetrisable Borcherds matrix arises from some k-species over a finite field k.*

PROOF. Given s_i, let k_i/k be a field extension of degree s_i. We may assume that the indices are given by the integers $1, \dots, n$. Then, for $i < j$, take k_{ij}/k a field extension of degree $-s_i c_{ij}$ and $k_{ji} = 0$. Finally, let k_{ii}/k be a field extension of degree $s_i(1 - c_{ii}/s)$. In this way we obtain a k-species corresponding to the matrix C. □

There is a natural way to associate a valued graph to any symmetrisable generalised Cartan matrix. Given $C = D^{-1}B$ with rows and columns indexed by Q_0, we let Q_0 be the set of vertices, with vertex i having value s_i and draw valued edges

$$s_i \overset{|b_{ij}|}{\frac{\qquad}{}} s_j \quad \text{for } i \neq j, \text{ and} \quad s_i \bigcirc_{\, s_i - b_{ii}/2}$$

We usually omit the edge if it has value 0.

Note that our definitions are not the standard ones, but are simpler to draw and retain the necessary information. In particular, we have a bijection between valued graphs and pairs (D, B) such that $C = D^{-1}B$ is a finite symmetrisable Borcherds matrix.

Similarly we can associate to any k-species a valued quiver. The vertices correspond to the primitive idempotents, with vertex i having value s_i, and we draw valued arrows

$$s_i \overset{a_{ij}}{\underset{a_{ji}}{\rightleftarrows}} s_j \quad \text{for } i \neq j, \text{ and} \quad s_i \circlearrowleft s_i - b_{ii}/2$$

If each $s_i = 1$, then we can omit these numbers and replace an arrow with value m by m unvalued arrows. In this way we recover the quiver Q from the path algebra kQ.

Exercises 5.

(1) Deduce Corollary 1.13 directly from the projective resolution of Theorem 1.9 by applying the functor $\mathrm{Hom}(-, N)$.
(2) Given a quiver Q, show that there exists a fully faithful functor $\mathrm{Mod}\, kQ \to \mathrm{Mod}\, kQ'$ (restricting to $\mathrm{mod}\, kQ \to \mathrm{mod}\, kQ'$) for some other quiver Q' without oriented cycles (so that kQ' is finite dimensional).
(3) Use the Euler form to show the the category $\mathrm{mod}\,\Lambda$ is hereditary, where Λ is a k-species. [Hint. Recall the definition of $\mathrm{Ext}^2_\Lambda(M, N)$ in terms of pairs of exact sequences. The Euler form shows that if $0 \to N \to E \to M \to 0$ is a short exact sequence, then $\mathrm{Ext}^1_\Lambda(E, X) \to \mathrm{Ext}^1_\Lambda(N, X)$ is surjective for all X (by considering dimensions). We deduce that $\mathrm{Ext}^2_\Lambda(M, N) = 0$.]

6. An Example

As an example, let K/k be a field extension of degree n. Then K is naturally a k-K-bimodule, so we can form a k-species Λ using $\Lambda_0 := k \times K$ and $\Lambda_1 := K$ (as a k-K-bimodule). Then

$$\Lambda := \begin{pmatrix} k & K \\ 0 & K \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -n \\ 0 & n \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -n \\ -1 & 2 \end{pmatrix} \tag{6.1}$$

and the valued quiver of Λ is

$$\bullet \xrightarrow{(n,1)} \bullet$$

A representation is given by a k-vector space U, a K-vector space V and a K-linear map $\phi\colon U \otimes_k K \to V$. Now,

$$\mathrm{Hom}_K(U \otimes_k K, V) \cong \mathrm{Hom}_k(U, \mathrm{Hom}_K(K, V)) \cong \mathrm{Hom}_k(U, V \otimes_K K) \cong \mathrm{Hom}_k(U, V),$$

using the natural K-k-bimodule structure on K. Thus a representation can also be thought of as a k-vector space U, a K-vector space V and a k-linear map $\phi\colon U \to V$.

There exists a uniuqe indecomposable (up to isomorphism) of dimension vector $e_1 + e_2$. For, any such is given by a 1-dimensional k-subspace of K. Since we can act by K^*, we may assume that this has basis vector $1 \in K$.

Similarly, there are no indecomposable representations of dimension vector $e_1 + 2e_2$. For, any 1-dimensional k-subspace of K^2 has basis vector $(1, 0)^t$ up to the action of $\mathrm{GL}_2(K)$.

Now consider indecomposable representations of dimension vector $2e_1 + e_2$; that is, a 2-dimensional k-subspace of K — so $n \geq 2$. We can view $\phi\colon k^2 \to K$ as given by a pair (x, y), where $x, y \in K$ are linearly independent over k, thus a basis for $\mathrm{Im}(\phi)$. Isomorphisms are given via

$$\begin{array}{ccc} k^2 & \xrightarrow{(x',y')} & K \\ {\scriptstyle \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle z} \\ k^2 & \xrightarrow{(x,y)} & K \end{array} \quad \text{where} \quad x' = \frac{ax + cy}{z}, \quad y' = \frac{bx + dy}{z}. \tag{6.2}$$

Since $x, y \in K$ are linearly dependent over k, we see that every indecomposable representation is isomorphic to one given by a pair $(x, 1)$ for some $x \in K \setminus k$. Moreover, the pairs $(x, 1)$ and $(x', 1)$ correspond to isomorphic representations if and only if there exists an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $x' = (ax + c)/(bx + d)$.

Thus we are interested in this "Möbius transformation" on $K \setminus k$, sending $x \mapsto (ax + c)/(bx + d)$.

Consider the endomorphism ring of the indecomposable representation corresponding to $(1, x)$. We see that this is given by those pairs $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) \in \mathbb{M}_2(k) \times K$ such that $z = bx + d$ and $zx = ax + c$; that is, $bx^2 + (d - a)x - c = 0$. If $1, x, x^2$ are linearly dependent over k, then the endomorphism ring is isomorphic to $k[x]$, a field extension of k of degree 2. Otherwise the endomorphism ring is simply k.

Furthermore, the group $\mathrm{GL}_2(k)$ has $q(q-1)(q^2-1)$ elements, and the stabiliser under the Möbius transformation has size $q - 1$ if $1, x, x^2$ are linearly independent over k, or else $q^2 - 1$ if they are linearly dependent.

Case $n = 2$.

In this case, $1, x, x^2$ are always linearly dependent over k, so the endomorphism ring is always isomorphic to K. Thus there are $q(q - 1)$ pairs of the form $(y, 1)$ in the orbit of $(x, 1)$. Hence all points $y \in K \setminus k$ occur and there is precisly one indecomposable representation up to isomorphism and its endomorphism ring is K.

Case $n = 3$.

In this case, there is no subfield $k \subset L \subset K$ with $[L : k] = 2$. Hence $1, x, x^2$ must be linearly independent and any indecomposable has endomorphism ring k. There are now $q(q^2-1)$ pairs of the form $(y, 1)$ in the orbit of x, hence all points $y \in K \setminus k$ occur. There is thus a unique indecomposable and its endomorphism ring is k.

Case $n = 4$.

In this case there is a unique subfield $k \subset L \subset K$ with $[L : k] = 2$. Any $x \in L \setminus k$ gives rise to an indecomposable with endomorphism ring L. There are $q(q-1)$ points in its orbit, which thus covers all elements in $L \setminus k$. Thus this indecomposable is unique up to isomorphism.

For all other points $x \in K \setminus L$, the corresponding indecomposable has endomorphism ring k, hence each orbit under the Möbius transformation has size $q(q^2 - 1)$. Therefore there are q orbits, so q isomorphism classes of indecomposables.

CHAPTER 2

Ringel-Hall Algebras

1. Hall Numbers and Ringel-Hall Algebras

In this section, we define the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ for the category $\mathcal{A} = \operatorname{mod}\Lambda$, where Λ is a species over some finite field k with q elements.

For three modules M, N and E we define

$$\mathcal{P}^E_{MN} := \{(f,g) : 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \text{ exact}\} \quad \text{and} \quad P^E_{MN} := |\mathcal{P}^E_{MN}|. \tag{1.1}$$

Note that P^E_{MN} is a finite number since both $\operatorname{Hom}(N,E)$ and $\operatorname{Hom}(E,M)$ are finite sets.

We also define

$$\mathcal{F}^E_{MN} := \{U \leq E : E/U \cong M, U \cong N\} \quad \text{and} \quad F^E_{MN} := |\mathcal{F}^E_{MN}|. \tag{1.2}$$

The F^E_{MN} are called Hall numbers.

There is a natural map $\mathcal{P}^E_{MN} \to \mathcal{F}^E_{MN}$ sending (f,g) to the submodule $\operatorname{Im}(f) = \operatorname{Ker}(g)$. The fibres are given by the natural action of $\operatorname{Aut}(M) \times \operatorname{Aut}(N)$ on $\mathcal{P}^E_{MN}$, and this action is free.

For any object X, the automorphism group $\operatorname{Aut}(X) \subset \operatorname{End}(X)$ is finite. We write

$$a_X := |\operatorname{Aut}(X)|. \tag{1.3}$$

The above considerations immediately give the following lemma.

Lemma 2.1. $F^E_{MN} = P^E_{MN}/a_M a_N$.

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is the free abelian group with basis $u_{[X]}$ parameterised by the isomorphism classes of objects in $\mathcal{A}$ and with multiplication

$$u_{[M]}u_{[N]} := \sum_{[E]} F^E_{MN} u_{[E]} = \sum_{[E]} \frac{P^E_{MN}}{a_M a_N} u_{[E]}. \tag{1.4}$$

We note that this sum is finite since $\operatorname{Ext}^1(M,N)$ is a finite set. Also, we are implicitly using the fact that $\mathcal{A}$ is essentially small, so that the basis of $\mathcal{H}(\mathcal{A})$ is a set.

We shall often abuse notation and write u_X for $u_{[X]}$. Similarly, although most summations will be over isomorphism classes of objects, we shall write $\sum_E$ for $\sum_{[E]}$.

Lemma 2.2 (Ringel). *The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is an associative algebra with unit u_0 corresponding to the zero module. In particular*

$$\sum_X F^X_{LM} F^E_{XN} = \sum_X F^E_{LX} F^X_{MN}$$

for all objects L, M, N and E.

Proof. Consider the products $u_L(u_M u_N)$ and $(u_L u_M)u_N$. We have

$$u_L(u_M u_N) = \sum_X F^X_{MN} u_L u_X = \sum_{X,E} F^E_{LX} F^X_{MN} u_E,$$
$$(u_L u_M)u_N = \sum_X F^X_{LM} u_X u_N = \sum_{X,E} F^X_{LM} F^E_{XN} u_E.$$

Comparing coefficients we see that the multiplication is associative if and only if we have the identity

$$\sum_X F^X_{LM} F^E_{XN} = \sum_X F^E_{LX} F^E_{MN} \quad \text{for all } L,\, M,\, N \text{ and } E.$$

We can rewrite this in terms of the numbers P^E_{MN} to get

$$\sum_X \frac{P^X_{LM} P^E_{XN}}{a_X} = \sum_X \frac{P^E_{LX} P^X_{MN}}{a_X} \quad \text{for all } L,\, M,\, N \text{ and } E.$$

We will show that there is a bijection

$$\coprod_X \frac{\mathcal{P}^X_{LM} \times \mathcal{P}^E_{XN}}{\mathrm{Aut}(X)} \longleftrightarrow \coprod_Y \frac{\mathcal{P}^E_{LY} \times \mathcal{P}^Y_{MN}}{\mathrm{Aut}(Y)},$$

where the action of $\mathrm{Aut}(X)$ on $\mathcal{P}^X_{LM} \times \mathcal{P}^E_{XN}$ is given via

$$\xi \cdot ((a,b),(f,g)) := ((\xi a, b\xi^{-1}),(f,\xi g)).$$

Note that this action is free, since g is an epimorphism. Similarly for the action of $\mathrm{Aut}(Y)$ on $\mathcal{P}^E_{LY} \times \mathcal{P}^Y_{MN}$.

There is a natural map

$$\mathcal{P}^X_{LM} \times \mathcal{P}^E_{XN} \longrightarrow \coprod_Y \frac{\mathcal{P}^E_{LY} \times \mathcal{P}^Y_{MN}}{\mathrm{Aut}(Y)}$$

given by the pull-back construction. This is well-defined since the pull-back is uniquely determined up to isomorphism. We can draw this in a commutative diagram as $((a,b),(f,g)) \mapsto ((a',b'),(f',g'))$, where

$$\begin{array}{ccccc}
N & = & N & & \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\
Y & \xrightarrow{a'} & E & \xrightarrow{b'} & L \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \| \\
M & \xrightarrow{a} & X & \xrightarrow{b} & L
\end{array}$$

In fact, this map only depends on the pair $((a,b),(f,g))$ up to the action of $\mathrm{Aut}(X)$, so we obtain an induced map

$$\coprod_X \frac{\mathcal{P}^X_{LM} \times \mathcal{P}^E_{XN}}{\mathrm{Aut}(X)} \longrightarrow \coprod_Y \frac{\mathcal{P}^E_{LY} \times \mathcal{P}^Y_{MN}}{\mathrm{Aut}(Y)}.$$

There is a map in the other direction induced by taking the push-out, and these constructions are easily seen to be mutual inverses. In other words, the square

$$\begin{array}{ccc}
Y & \xrightarrow{a'} & E \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\
M & \xrightarrow{a} & X
\end{array}$$

is both a push-out and a pull-back, or a homotopy Cartesian square.

It is clear that u_0 is the identity for this multiplication. □

In fact, this proof works much more generally. We only need $\mathcal{A}$ to be an essentially small, finitary, exact category: that is, an exact category for which $\mathrm{Hom}(M,N)$ and $\mathrm{Ext}^1(M,N)$ are finite abelian groups and such that the isomorphism classes of objects form a set.

For example, we may take $\mathcal{A}$ to be the finite length modules over some ring with only finitely many elements, since each such module will have only finitely many elements [Ringel]. We could also take the category of coherent sheaves over some projective k-scheme [Schiffmann].

We note that iterated multiplications can be expressed using filtrations. Given objects $M_1,\dots,M_r$ and E, define

$$\begin{aligned}\mathcal{F}^E_{M_1\cdots M_r} &:= \{0=U_{r+1}\subset U_r\subset\cdots\subset U_1=E : U_i/U_{i+1}\cong M_i\},\\ F^E_{M_1\cdots M_r} &:= |\mathcal{F}^E_{M_1\cdots M_r}|.\end{aligned} \tag{1.5}$$

Then

$$u_{M_1}\cdots u_{M_r}=\sum_E F^E_{M_1\cdots M_r}u_E. \tag{1.6}$$

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is naturally graded by the Grothendieck group $K(\mathcal{A})$, where

$$\mathcal{H}(\mathcal{A})_\alpha=\bigoplus_{\bar{M}=\alpha}\mathbb{Z}u_M. \tag{1.7}$$

Then $\mathcal{H}$ acquires the structure of a graded algebra; that is, $\mathcal{H}_{\underline{d}}\mathcal{H}_{\underline{e}}\subset\mathcal{H}_{\underline{d}+\underline{e}}$.

Finally we relate Hall numbers to counting extension classes. Given objects M, N and E write $\mathrm{Ext}^1_\Lambda(M,N)_E\subset\mathrm{Ext}^1(M,N)$ for the set of all classes of extensions of M by N which are isomorphic to E.

Proposition 2.3 (Riedtmann's Formula).

$$F^E_{MN}=\frac{|\,\mathrm{Ext}^1(M,N)_E|}{|\,\mathrm{Hom}(M,N)|}\cdot\frac{a_E}{a_Ma_N}.$$

Proof. Recall that there is a natural map $\mathcal{P}^E_{MN}\to\mathrm{Ext}^1(M,N)_E$ sending (f,g) to the class of the corresponding extension, which we may denote by $[(f,g)]$. The fibre over a given class $[(f,g)]$ consits of those pairs $(f',g')=(\theta f,g\theta^{-1})$ for some $\theta\in\mathrm{Aut}(E)$. Pictorially this is given by

$$\begin{array}{ccccccccc}0&\longrightarrow&N&\xrightarrow{f}&E&\xrightarrow{g}&M&\longrightarrow&0\\ &&\|&&\downarrow{\scriptstyle\theta}&&\|&&\\ 0&\longrightarrow&N&\xrightarrow{f'}&E&\xrightarrow{g'}&M&\longrightarrow&0\end{array}$$

Thus we have an action of $\mathrm{Aut}(E)$ on $\mathcal{P}^E_{MN}$, with quotient precisely $\mathrm{Ext}^1(M,N)_E$. Thus we need to describe the stabiliser in $\mathrm{Aut}(E)$ of some point $(f,g)\in\mathcal{P}^E_{MN}$.

The stabiliser of (f,g) is the set of automorphisms θ such that $g\theta=g$ and $\theta f=f$. Since $(\theta-1)f=0$, we see from the long exact sequence for $\mathrm{Hom}(-,E)$ that there exists a unique $\phi\in\mathrm{Hom}(M,E)$ satisfying $\phi g=\theta-1$. Now $0=g(\theta-1)=g\phi g$. Since g is an epimorphism, we deduce that $g\phi=0$. Applying $\mathrm{Hom}(M,-)$ shows that there exists a unique $\psi\in\mathrm{Hom}(M,N)$ such that $\phi=f\psi$, hence that $\theta=1+f\psi g$.

We have shown that there exists an injective map $\mathrm{Stab}(f,g)\to\mathrm{Hom}(M,N)$ sending θ to the unique ψ such that $\theta=1+f\psi g$. This map is also surjective, since if $\psi \in \operatorname{Hom}(M,N)$, then $f\psi g \in \operatorname{End}(E)$ satisfies $(f\psi g)^2 = 0$, so that $1 + f\psi g \in \operatorname{Aut}(E)$. Hence the stabiliser of any point has size $|\operatorname{Hom}(M,N)|$.

We deduce that $P^E_{MN} = |\operatorname{Ext}^1(M,N)| \frac{a_E}{|\operatorname{Hom}(M,N)|}$. □

2. The Coalgebra Structure

Since the multiplication is given by taking two objects and forming extensions of these, it is natural to ask the dual question about breaking an object into a submodule and its factor module. We need to be careful here, however, since it is not true in general that an object has only finitely many subobjects.

Define a comultiplication $\Delta\colon \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ on the free abelian group $\mathcal{H}(\mathcal{A})$ via

$$\Delta(u_E) := \sum_{M,N} F^E_{MN} \frac{a_M a_N}{a_E} u_M \otimes u_N = \sum_{M,N} \frac{P^E_{MN}}{a_E} u_M \otimes u_N. \tag{2.1}$$

We see that this notion is really dual to the multiplication, where we used the structure constants $\frac{P^E_{MN}}{a_M a_N}$.

Lemma 2.4. *The Ringel-Hall coalgebra is coassociative with counit* $\epsilon(u_M) = \delta_{M0}$.

Proof. We have

$$(\Delta \otimes 1)\Delta(u_E) = \sum_{N,X} \frac{P^E_{XN}}{a_E} \Delta(u_X) \otimes u_N = \sum_{L,M,N,X} \frac{P^X_{LM} P^E_{XN}}{a_X a_E} u_L \otimes u_M \otimes u_N,$$

$$(1 \otimes \Delta)\Delta(u_E) = \sum_{L,X} \frac{P^E_{LX}}{a_E} u_L \otimes \Delta(u_X) = \sum_{L,M,N,X} \frac{P^E_{LX} P^X_{MN}}{a_X a_E} u_L \otimes u_M \otimes u_N.$$

Thus the coassociativity of Δ follows from the same identity used to prove the associativity of Hall multiplication:

$$\sum_X \frac{P^X_{LM} P^E_{XN}}{a_X} = \sum_X \frac{P^E_{LX} P^X_{MN}}{a_X}.$$

For the counit, we note the identity

$$(\epsilon \otimes 1)\Delta(u_E) = \sum_{M,N} \frac{P^E_{MN}}{a_E} \epsilon(u_M) \otimes u_N = \frac{P^E_{0E}}{a_E} 1 \otimes u_E = 1 \otimes u_E,$$

since $P^E_{0E} = a_E$. Similarly $(1 \otimes \epsilon)\Delta(u_E) = u_E \otimes 1$. □

In fact, we see from the proof that the comultiplication can also be defined for exact categories satisfying some finiteness condition: namely that each object has only finitely many subobjects. This clearly holds for the category of finite length modules over a finite ring, but fails for the category of coherent sheaves over a projective scheme. In this case, one needs to work harder, using a completion of the Hall alegbra [Schiffmann].

The comultiplication respects the grading on $\mathcal{H}(\mathcal{A})$ given by the Grothendieck group $K(\mathcal{A})$:

$$\Delta\colon \mathcal{H}_{\underline{d}} \to \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \mathcal{H}_{\underline{e}} \otimes \mathcal{H}_{\underline{e}'}.$$

Also, if S is simple, then u_S is a primitive element

$$\Delta(u_S) = u_S \otimes 1 + 1 \otimes u_S,$$

but the converse is almost never true. In fact, it is important to be able to find the primitive elements [Sevenhant-Van den Bergh].

We can again relate powers of the comultiplication to numbers of filtrations via the formula

$$\Delta^r(u_E) = \sum_{M_1,\cdots,M_r} F^E_{M_1\cdots M_r} \frac{a_{M_1}\cdots a_{M_r}}{a_E} u_{M_1} \otimes \cdots \otimes u_{M_r}. \tag{2.2}$$

3. Examples

One of the simplest examples is when $\mathcal{A}$ is the category of vector spaces over some finite field k with q elements. This is equivalent to the category $\operatorname{mod} k = \operatorname{mod} kQ$ for the quiver consisting of a single vertex.

Write u_m, $m \geq 0$, for the basis element $u_{[k^m]}$ in $\mathcal{H}(\mathcal{A})$. Then

$$u_m u_n = F^{m+n}_{mn} u_{m+n}$$

and the Hall number F^{m+n}_{mn} counts the number of n-dimensional subspaces of k^{m+n}, so equals the number of points in the Grassmannian $\mathrm{Gr}\binom{m+n}{n}$ over k. Thus

$$F^{m+n}_{mn} = |\mathrm{Gr}\tbinom{m+n}{n}| = \begin{bmatrix} m+n \\ n \end{bmatrix}_+ := \tfrac{[m+n]_+!}{[m]_+![n]_+!},$$

where we have used the quantum numbers

$$[m]_+ := \frac{q^m-1}{q-1} \quad \text{and} \quad [m]_+! := [m]_+[m-1]_+\cdots[1]_+.$$

Note that $u_m = \frac{u_1^m}{[m]_+!}$ is a divided power and that F^{m+n}_{mn} is given by a "universal polynomial" in $\mathbb{Z}[T]$, evaluated at $T = q = |k|$.

We also wish to consider the comultiplication. We first note that

$$a_m = |\operatorname{Aut}(k^m)| = |\operatorname{GL}_m(q)| = (q^m-1)(q^m-q)\cdots(q^m-q^{m-1}) = q^{\binom{m}{2}}(q-1)^m[m]_+!$$

This is again given by a universal polynomial. Now,

$$\begin{aligned} P^{m+n}_{mn} &= F^{m+n}_{mn} a_m a_n = q^{\binom{m}{2}+\binom{n}{2}}(q-1)^{m+n}[m+n]_+! \\ &= q^{\binom{m+n}{2}-mn}(q-1)^{m+n}[m+n]_+! = a_{m+n}q^{-mn}. \end{aligned}$$

Thus

$$\Delta(u_r) = \sum_{m+n=r} q^{-mn} u_m \otimes u_n.$$

We observe that $\mathcal{H}(\mathcal{A})$ is both commutative and cocommutative. In fact, this holds whenever $\mathcal{A}$ is a semisimple category.

Finally we wish to compare the multiplication and comultiplication. We first calculate

$$\Delta(u_r u_s) = \begin{bmatrix} r+s \\ r \end{bmatrix}_+ \Delta(u_{r+s}) = \sum_{m+n=r+s} q^{-mn} \begin{bmatrix} m+n \\ r \end{bmatrix}_+ u_m \otimes u_n,$$

$$\Delta(u_r)\Delta(u_s) = \sum_{\substack{a+b=r \\ c+d=s}} q^{-ab-cd} u_a u_c \otimes u_b u_d = \sum_{\substack{m+n=r+s \\ a,b,c,d}} q^{-ab-cd} \begin{bmatrix} m \\ a \end{bmatrix}_+ \begin{bmatrix} n \\ b \end{bmatrix}_+ u_m \otimes u_n,$$

where the latter sum is over those a, b, c and d such that

$$a+b=r, \quad c+d=s, \quad a+c=m, \quad b+d=r.$$

In order to compare these formulae, we need a lemma, which is the quantum analogue of the classical formula

$$\binom{m+n}{r}=\sum_{a+b=r}\binom{m}{a}\binom{n}{b}.$$

LEMMA 2.5. *The following holds for quantum binomial coefficients:*

$$\begin{bmatrix} m+n \\ r \end{bmatrix}_+ = \sum_{a+b=r} q^{b(m-a)} \begin{bmatrix} m \\ a \end{bmatrix}_+ \begin{bmatrix} n \\ b \end{bmatrix}_+.$$

PROOF. The proof for ordinary binomial coefficients runs as follows. We divide the set of size $m+n$ into a set of size m and a set of size n. To choose a subset of size r is then to choose a decomposition $r=a+b$ together with a subset of size a from the set of size m and a subset of size b from the set of size n.

We now emulate this proof, using the philosophy that choosing r points from a set of size n corresponds to choosing an r-dimensional subspace of a vector space of dimension n. (See John Baez's Stuff `http://math.ucr.edu/home/baez/week184.html` .)

We fix a short exact sequence

$$0 \to k^m \to k^{m+n} \xrightarrow{p} k^n \to 0$$

and a decomposition $r=a+b$. This defines a closed subscheme $G_{a,b}$ of the Grassmannian $\mathrm{Gr}\binom{m+n}{r}$ via

$$G_{a,b} := \{U \in \mathrm{Gr}\tbinom{m+n}{r} : \dim p(U) = b\}.$$

Then $G_{a,b}$ is naturally a vector bundle over the product $\mathrm{Gr}\binom{m}{a} \times \mathrm{Gr}\binom{n}{b}$, where $U \mapsto (U \cap k_m, p(U))$. The fibre over some point (V,W) is canonically isomorphic to $\mathrm{Hom}(W, k^m/V)$. Counting points now completes the proof. □

Using this result, together with the identity $-mn+b(m-a)=-ab-cd-ad$, we see that

$$\begin{aligned} \Delta(u_r u_s) &= \sum_{\substack{m+n=r+s \\ a,b,c,d}} q^{-ab-cd-ad} \begin{bmatrix} m \\ a \end{bmatrix}_+ \begin{bmatrix} n \\ b \end{bmatrix}_+ u_m \otimes u_n \\ \Delta(u_r)\Delta(u_s) &= \sum_{\substack{m+n=r+s \\ a,b,c,d}} q^{-ab-cd} \begin{bmatrix} m \\ a \end{bmatrix}_+ \begin{bmatrix} n \\ b \end{bmatrix}_+ u_m \otimes u_n, \end{aligned} \tag{3.1}$$

where a, b, c and d satisfy the same relations as before. Hence even in this simple example, the comultiplication is not an algebra homomorphism.

Our second example involves $\mathcal{A} = \mathrm{mod}\, kQ$, where k is a finite field with q elements and Q is the quiver of type $\mathbb{A}_2$

$$1 \longrightarrow 2$$

We have indecomposables S_1, S_2 and X, where X is both projective and injective. Write u_i for u_{S_i}. We calculate some monomials in the u_i. We have

$$u_2u_1 = \sum_E F^E_{S_2S_1}u_E = u_{S_1\oplus S_2}.$$

For, any exact sequence

$$0 \to S_1 \to E \to S_2 \to 0$$

must split, since S_2 is projective. Since there are no homomorphisms between non-isomorphic simple modules, the Hall number $F^{S_1\oplus S_2}_{S_2S_1}$ equals 1.

On the other hand,

$$u_1u_2 = \sum_E F^E_{S_1S_2}u_E = u_X + u_{S_1\oplus S_2}.$$

For, there are only two non-isomorphic modules with dimension vector $e_1 + e_2$, namely X and $S_1 \oplus S_2$. Both of these contain a unique submodule isomorphic to S_2, and the cokernel is necessarily S_1. Thus both Hall numbers are 1.

We know that $\mathrm{End}(S_i) = k$ and $a_i = q - 1$. Thus

$$u_i^2 = [2]_+ u_{S_i^2} = (q+1)u_{S_i^2}.$$

Similar considerations yield the following tables of Hall numbers $F^E_{S_iS_jS_k}$, where the triples (i, j, k) label the rows and the modules E label the columns:

	$S_1 \oplus X$	$S_1^2 \oplus S_2$
$(1,1,2)$	$q+1$	$q+1$
$(1,2,1)$	1	$q+1$
$(2,1,1)$	0	$q+1$

	$X \oplus S_2$	$S_1 \oplus S_2^2$
$(1,2,2)$	$q+1$	$q+1$
$(2,1,2)$	1	$q+1$
$(2,2,1)$	0	$q+1$

For example, we have that

$$u_1^2u_2 = (q+1)u_{S_1\oplus X} + (q+1)u_{S_1^2\oplus S_2} \quad \text{and} \quad u_2u_1u_2 = u_{X\oplus S_2} + (q+1)u_{S_1\oplus S_2^2}.$$

From these tables we obtain the quantum Serre relations

$$u_1^2u_2 - (q+1)u_1u_2u_1 + qu_2u_1^2 = 0 \quad \text{and} \quad u_1u_2^2 - (q+1)u_2u_1u_2 + qu_2^2u_1 = 0. \tag{3.2}$$

We next consider the comultiplication. We have already remarked that simple objects must be primitive, so consider the comultiplication applied to $S_1 \oplus S_2$ and to X. We have

$$\begin{aligned} \Delta(u_{S_1\oplus S_2}) &= u_{S-1\oplus S_2} \otimes 1 + u_1 \otimes u_2 + u_2 \otimes u_1 + 1 \otimes u_{S_1\oplus S_2} \\ \Delta(u_X) &= u_X \otimes 1 + (q-1)u_1 \otimes u_2 + 1 \otimes u_X, \end{aligned} \tag{3.3}$$

where we have used that $\mathrm{End}(X) = k$ and $a_X = q - 1$.

We note that $\mathcal{H}(\mathcal{A})$ is neither commutative nor cocommutative. This is generally the case for Ringel-Hall algebras.

We now try to relate the multiplication and comultiplication. Writing $u_{S_1\oplus S_2} = u_2u_1$ as a product, we have that

$$\begin{aligned} \Delta(u_2)\Delta(u_1) &= (u_2 \otimes 1 + 1 \otimes u_2)(u_1 \otimes 1 + 1 \otimes u_1) \\ &= u_2u_1 \otimes 1 + u_2 \otimes u_1 + u_1 \otimes u_2 + 1 \otimes u_2u_1 = \Delta(u_2u_1). \end{aligned} \tag{3.4}$$

If we consider the product

$$u_1u_2 = u_X + u_{S_1\oplus S_2} = u_X + u_2u_1,$$

however, we obtain

$$\begin{aligned} \Delta(u_1)\Delta(u_2) &= u_1u_2 \otimes 1 + u_1 \otimes u_2 + u_2 \otimes u_1 + 1 \otimes u_1u_2, \\ \Delta(u_1u_2) = \Delta(u_X) + \Delta(u_2u_1) &= u_1u_2 \otimes 1 + qu_1 \otimes u_2 + u_2 \otimes u_1 + 1 \otimes u_1u_2. \end{aligned} \tag{3.5}$$

We again see that Δ is not an algebra homomorphism.

4. Green's Formula and the Hopf Algebra Structure

By analogy to quantum groups (which we have not yet introduced), one does not expect that the multiplication and comultiplication are directly compatible, but rather that one needs to introduce a twist. Define a new multiplication on the tensor product $\mathcal{H} \otimes \mathcal{H}$ by

$$(u_A \otimes u_B) \cdot (u_C \otimes u_D) := q^{-\langle A,D \rangle} u_A u_C \otimes u_B u_D. \tag{4.1}$$

Consider our examples again.

The first example, where $\mathcal{A}$ is the category of k-vector spaces, yields

$$\begin{aligned} \Delta(u_r) \cdot \Delta(u_s) &= \sum_{\substack{a+b=r \\ c+d=s}} q^{-ab-cd} (u_a \otimes u_b) \cdot (u_c \otimes u_d) \\ &= \sum_{\substack{a+b=r \\ c+d=s}} q^{-ab-cd-ad} u_a u_c \otimes u_b u_d = \Delta(u_r u_s), \end{aligned} \tag{4.2}$$

where we have used that $\mathcal{A}$ is a semisimple category, so that

$$\langle k^a, k^d \rangle = \dim \operatorname{Hom}(k_a, k_d) = ad.$$

The second example yields

$$\begin{aligned} \Delta(u_2) \cdot \Delta(u_1) &= (u_2 \otimes 1 + 1 \otimes u_2) \cdot (u_1 \otimes 1 + 1 \otimes u_1) \\ &= u_2u_1 \otimes 1 + u_1 \otimes u_2 + u_2 \otimes u_1 + 1 \otimes u_2u_1 = \Delta(u_2u_1) \\ \Delta(u_1) \cdot \Delta(u_2) &= (u_1 \otimes 1 + 1 \otimes u_1) \cdot (u_2 \otimes 1 + 1 \otimes u_2) \\ &= u_1u_2 \otimes 1 + qu_1 \otimes u_2 + u_2 \otimes u_1 + 1 \otimes u_1u_2 = \Delta(u_1u_2), \end{aligned} \tag{4.3}$$

where we have used that

$$\langle S_1, S_2 \rangle = -\dim \operatorname{Ext}^1(S_1, S_2) = -1 \quad \text{and} \quad \langle S_2, S_1 \rangle = -\dim \operatorname{Ext}^1(S_2, S_1) = 0.$$

Hence in both cases Δ becomes an algebra homomorphism. The main theorem of this chapter says that this is always the case.

THEOREM 2.6 (Green). *The comultiplication is an algebra homomorphism with respect to this twisted multiplication on $\mathcal{H} \otimes \mathcal{H}$.*

Consider what this means for objects M and N. On the one hand we have

$$\Delta(u_M u_N) = \sum_E \frac{P^E_{MN}}{a_M a_N} \Delta(u_E) = \sum_{X,Y} \sum_E \frac{P^E_{MN} P^E_{XY}}{a_M a_N a_E} u_X \otimes u_Y, \tag{4.4}$$

whreas on the other hand we have

$$\Delta(u_M)\cdot\Delta(u_N) = \sum_{A,B,C,D} \frac{P^M_{AB}P^N_{CD}}{a_M a_N}(u_A\otimes u_B)\cdot(u_C\otimes u_D) \tag{4.5}$$

$$= \sum_{A,B,C,D} q^{-\langle A,D\rangle}\frac{P^M_{AB}P^N_{CD}}{a_M a_N} u_A u_C\otimes u_B u_D$$

$$= \sum_{X,Y}\sum_{A,B,C,D} q^{-\langle A,D\rangle}\frac{P^M_{AB}P^N_{CD}P^X_{AC}P^Y_{BD}}{a_M a_N a_A a_B a_C a_D} u_X\otimes u_Y. \tag{4.6}$$

The theorem is therefore equivalent to the following proposition.

Proposition 2.7 (Green's Formula). *For all M, N, X and Y we have the identity*

$$\sum_E \frac{P^E_{MN}P^E_{XY}}{a_E} = \sum_{A,B,C,D} q^{-\langle A,D\rangle}\frac{P^M_{AB}P^N_{CD}P^X_{AC}P^Y_{BD}}{a_A a_B a_C a_D}.$$

Rewriting this in terms of Hall numbers we have

$$\sum_E \frac{F^E_{MN}F^E_{XY}}{a_E} = \sum_{A,B,C,D} q^{-\langle A,D\rangle} F^M_{AB}F^N_{CD}F^X_{AC}F^Y_{BD}\frac{a_A a_B a_C a_D}{a_M a_N a_X a_Y}.$$

The rest of this section will be devoted to proving this formula. We fix representations M, N, X and Y.

Recall that the associativity and coassociativity of $\mathcal{H}$, Lemmas 2.2 and 2.4, were proved by considering push-out/pull-back diagrams.

In a similar vein, we reformulate Green's Formula in terms of 3×3 exact commutative diagrams of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \xrightarrow{\delta_Y} & Y & \xrightarrow{\beta_Y} & B & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \delta_N} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \beta_M} & & \\
0 & \longrightarrow & N & \xrightarrow{\nu} & E & \xrightarrow{\mu} & M & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \gamma_N} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \alpha_M} & & \\
0 & \longrightarrow & C & \xrightarrow{\gamma_X} & X & \xrightarrow{\alpha_X} & A & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.7}$$

For convenience, we also set

$$\begin{aligned}\alpha &:= \alpha_M\mu = \alpha_X\xi, & \beta &:= \beta_M\beta_Y = \mu\eta,\\ \delta &:= \eta\delta_Y = \nu\delta_N, & \gamma &:= \gamma_X\gamma_N = \xi\nu.\end{aligned} \tag{4.8}$$

The left hand side of Green's Formula then corresponds to counting "crosses", given by ignoring the corners, whereas the right hand side is given by counting "frames", given by ignoring the centre.

Define $\mathcal{D}(A,B,C,D:E)$ to be the set of all exact commutative 3×3 diagrams of the above form. More precisely, $\mathcal{D}$ is the set of all morphisms which fit into such a diagram.

We note that

$$\underline{e} := \underline{\dim}\, E = \underline{\dim}(M\oplus N) = \underline{\dim}(X\oplus Y) = \underline{\dim}(A\oplus B\oplus C\oplus D). \tag{4.9}$$

We first wish to count the number of crosses. In particular, we need to know how to construct the corner objects from a given cross.

LEMMA 2.8. *Consider a partial exact commutative 3×3 diagram of the form*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & D & \longrightarrow & Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We can complete this to a full 3×3 diagram if and only if the top left square is a pull-back and the bottom right square is a push-out.

PROOF. Suppose that D is a pull-back and A a push-out. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y\oplus N & \xrightarrow{\left(\begin{smallmatrix}\eta & 0\\ 0 & \nu\end{smallmatrix}\right)} & E^2 & \xrightarrow{\left(\begin{smallmatrix}\xi & 0\\ 0 & \mu\end{smallmatrix}\right)} & X\oplus M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\eta,\nu)} & & \downarrow{\scriptstyle (1,1)} & & \downarrow & & \\
0 & \longrightarrow & E & = & E & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Since D is a pull-back, the Snake Lemma gives that $0\to D \xrightarrow{\delta} E \xrightarrow{\left(\begin{smallmatrix}\xi\\ -\mu\end{smallmatrix}\right)} X\oplus M$ is exact. Since A is a push-out, we have by definition that A is the cokernel of the right-most map. This shows that the first sequence below is exact. The exactness of the second sequence is shown similarly.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & D & \xrightarrow{\delta} & E & \xrightarrow{\left(\begin{smallmatrix}\xi\\ -\mu\end{smallmatrix}\right)} & X\oplus M & \xrightarrow{(\alpha_X,\alpha_M)} & A & \longrightarrow & 0 \\
0 & \longrightarrow & D & \xrightarrow{\left(\begin{smallmatrix}\delta_Y\\ -\delta_N\end{smallmatrix}\right)} & Y\oplus N & \xrightarrow{(\eta,\nu)} & E & \xrightarrow{\alpha} & A & \longrightarrow & 0.
\end{array} \tag{4.10}$$

Now consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)} & Y\oplus N & \xrightarrow{(1,0)} & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle (\eta,\nu)} & & \downarrow{\scriptstyle \mu\eta} & & \\
0 & \longrightarrow & N & \xrightarrow{\nu} & E & \xrightarrow{\mu} & M & \longrightarrow & 0
\end{array} \tag{4.11}$$

If D is a pull-back and A a push-out, then applying the Snake Lemma to (4.11) and using (4.10) yields the exact sequence

$$0 \longrightarrow D \xrightarrow{\delta} Y \xrightarrow{\mu\eta} M \xrightarrow{\alpha_M} A \longrightarrow 0.$$

For, the only non-trivial part is to determine the map $x\colon M \to A$. We know that $x\mu = \alpha = \alpha_M\mu$, so the surjectivity of μ gives $x = \alpha_M$. Factoring $\mu\eta$ through its image B provides a completion of the top right square. Dually we can find C, and hence a completion to a full 3×3 diagram.

Conversely, suppose that we can complete to a full 3×3 diagram. Then $\mu\eta = \beta = \beta_Y\beta_M$ and so we have an exact sequence

$$0 \longrightarrow D \xrightarrow{\delta_Y} Y \xrightarrow{\beta} M \xrightarrow{\alpha_M} A \longrightarrow 0.$$

Thus the Snake Lemma applied to (4.11) yields the exact sequence

$$0 \longrightarrow D \xrightarrow{\left(\begin{smallmatrix}\delta_Y\\ -\delta_N\end{smallmatrix}\right)} Y \oplus N \xrightarrow{(\eta,\nu)} E \xrightarrow{\alpha} A \longrightarrow 0.$$

For, the only non-trivial part is to determine the map $x\colon D \to N$. We know that $\nu x = \eta\delta_Y = \delta = \nu\delta_N$, so the injectivity of ν gives $x = \delta_N$. Hence the top left square is a pull-back. Dually for the bottom right square. □

Define $\mathcal{C}(A, B, C, D : E)$ to be the set of all crosses of the form

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & Y & & & & \\ & & & & \downarrow{\scriptstyle\eta} & & & & \\ 0 & \longrightarrow & N & \xrightarrow{\nu} & E & \xrightarrow{\mu} & M & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\xi} & & & & \\ & & & & X & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

such that D is the pull-back in the top left corner, A is the push-out in the bottom right corner, and B and C are the kernels/cokernels whose existence is ensured by the previous lemma.

Lemma 2.9. *The canonical map* $\mathcal{D}(A, B, C, D; E) \to \mathcal{C}(A, B, C, D; E)$ *is surjective, with fibres isomorphic to* $\mathrm{Aut}(A) \times \mathrm{Aut}(B) \times \mathrm{Aut}(C) \times \mathrm{Aut}(D)$.

Proof. Given a cross, we may form the pull-back in the top left corner, the push-out in the bottom right corner, and then complete the top right and bottom left corners using the previous lemma. Hence the map is surjective. Moreover, by construction, the fibres are given by the orbits of the canonical action of the group $\mathrm{Aut}(A) \times \mathrm{Aut}(B) \times \mathrm{Aut}(C) \times \mathrm{Aut}(D)$ on $\mathcal{D}(A, B, C, D; E)$. This is clear since all constructions are either kernels or cokernels, hence unique up to unique isomorphism. In particular, the group action is free. □

Next, define $\mathcal{F}(A, B, C, D)$ to be the set of all frames of the form

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & Y & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & N & & & & M & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

LEMMA 2.10. *Each frame yields a canonical element in* $\operatorname{Ext}^2(A, D)$ *as follows. We form the push-out of the top left corner and the pull-back of the bottom right corner to get*

$$\begin{array}{ccc}
D & \xrightarrow{\delta_Y} & Y \\
\big\downarrow{\scriptstyle \delta_N} & & \big\downarrow{\scriptstyle \eta'} \\
N & \xrightarrow{\nu'} & S
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
R & \xrightarrow{\mu'} & M \\
\big\downarrow{\scriptstyle \xi'} & & \big\downarrow{\scriptstyle \alpha_M} \\
X & \xrightarrow{\alpha_X} & A
\end{array}$$

Set $\alpha' := \alpha_M \mu' = \alpha_X \xi'$ *and* $\delta' := \nu' \delta_N = \eta' \delta_Y$. *Then there exists an exact sequence*

$$0 \longrightarrow D \xrightarrow{\delta'} S \xrightarrow{h} R \xrightarrow{\alpha'} A \longrightarrow 0$$

and h *is uniquely determined by the relation*

$$\begin{pmatrix} \mu' \\ \xi' \end{pmatrix} h(\eta', \nu') = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}.$$

PROOF. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\left(\begin{smallmatrix} \delta_Y \\ -\delta_N \end{smallmatrix}\right)} & Y \oplus N & \xrightarrow{(\eta', \nu')} & S & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \left(\begin{smallmatrix} \beta & 0 \\ 0 & \gamma \end{smallmatrix}\right)} & & \big\downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & R & \xrightarrow{\left(\begin{smallmatrix} \mu' \\ \xi' \end{smallmatrix}\right)} & M \oplus X & \xrightarrow{(\alpha_M, -\alpha_X)} & A & \longrightarrow & 0
\end{array}$$

Since we clearly have the exact sequence

$$0 \longrightarrow D^2 \xrightarrow{\left(\begin{smallmatrix} \delta_Y & 0 \\ 0 & \delta_N \end{smallmatrix}\right)} Y \oplus N \xrightarrow{\left(\begin{smallmatrix} \beta & 0 \\ 0 & \gamma \end{smallmatrix}\right)} M \oplus X \xrightarrow{\left(\begin{smallmatrix} \alpha_M & 0 \\ 0 & \alpha_X \end{smallmatrix}\right)} A^2 \longrightarrow 0,$$

we can apply the Snake Lemma to the above diagram to obtain the exact sequence

$$0 \longrightarrow D \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} D^2 \xrightarrow{(\delta', \delta')} S \xrightarrow{h} R \xrightarrow{\left(\begin{smallmatrix} \alpha' \\ \alpha' \end{smallmatrix}\right)} A^2 \xrightarrow{(1, -1)} A \longrightarrow 0.$$

This yields the exact sequence

$$0 \longrightarrow D \xrightarrow{\delta'} S \xrightarrow{h} R \xrightarrow{\alpha'} A \longrightarrow 0.$$

Also, by definition, the connecting homomorphism h satisfies

$$\begin{pmatrix} \mu' \\ \xi' \end{pmatrix} h(\eta', \nu') = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}.$$

Since $\binom{\mu'}{\xi'}$ is injective and (η', ν') surjective, this determines h uniquely. □

Lemma 2.11. *A frame can be completed to a 3×3 diagram if and only if the corresponding element of* $\mathrm{Ext}^2(A, D)$ *vanishes. In particular, every frame can be completed to a 3×3 diagram if and only if the category is hereditary.*

Proof. Recall that the exact sequence

$$0 \longrightarrow D \xrightarrow{\delta'} S \xrightarrow{h} R \xrightarrow{\alpha'} A \longrightarrow 0$$

vanishes in $\mathrm{Ext}^2(A, D)$ precisely when there exists an exact commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{\delta'} & S & \xrightarrow{\pi} & I & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow \iota & & \\ 0 & \longrightarrow & D & \xrightarrow{\delta} & E & \xrightarrow{g} & R & \longrightarrow & 0 \\ & & & & \downarrow \alpha & & \downarrow \alpha' & & \\ & & & & A & = & A & & \end{array} \tag{4.12}$$

where $h = \iota\pi$ is the factorisation of h via its image. In particular, the top right square is homotopy Cartesian.

Suppose that we can complete the frame to a 3×3 diagram. By the universal property of S and R we have unique maps $f\colon S \to E$ and $g\colon E \to R$ such that

$$f(\nu', \eta') = (\nu, \eta) \qquad \text{and} \qquad \begin{pmatrix} \mu' \\ \xi' \end{pmatrix} g = \begin{pmatrix} \mu \\ \xi \end{pmatrix}.$$

In particular, $f\delta' = f\eta'\delta_Y = \eta\delta_Y = \delta$ and similarly $\alpha' g = \alpha$. Moreover,

$$\begin{pmatrix} \mu' \\ \xi' \end{pmatrix} gf(\eta', \nu') = \begin{pmatrix} \mu \\ \xi \end{pmatrix} (\eta, \nu) = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix},$$

so that $gf = h$.

Applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{\binom{\delta_Y}{-\delta_N}} & Y \oplus N & \xrightarrow{(\eta', \nu')} & S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow (\eta,\nu) & & \downarrow f & & \\ 0 & \longrightarrow & 0 & \longrightarrow & E & = & E & \longrightarrow & 0 \end{array}$$

and using the following exact sequence from (4.10),

$$0 \longrightarrow D \xrightarrow{\binom{\delta_Y}{-\delta_N}} Y \oplus N \xrightarrow{(\eta,\nu)} E \xrightarrow{\alpha} A \longrightarrow 0,$$

gives that

$$0 \longrightarrow S \xrightarrow{f} E \xrightarrow{\alpha} A \longrightarrow 0$$

is exact. Similarly, the sequence

$$0 \longrightarrow D \xrightarrow{\delta} E \xrightarrow{g} R \longrightarrow 0$$

is exact and we obtain a diagram as in (4.12).

Conversely, suppose we have a diagram as in (4.12). We set

$$\nu := f\nu', \quad \mu := \mu' g, \quad \eta := f\eta', \quad \xi := \xi' g.$$

Then, from the definition of $h = \iota\pi = gf$, we have

$$\begin{pmatrix}\mu\\ \xi\end{pmatrix}(\eta,\nu) = \begin{pmatrix}\mu'\\ \xi'\end{pmatrix}h(\eta',\nu') = \begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}.$$

Hence we can fit E into a commutative 3×3 diagram. We need to show that the middle row and column are exact.

Since S is a push-out, we have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\delta_Y} & Y & \xrightarrow{\beta_Y} & B & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\delta_N} & & \downarrow{\scriptstyle\eta'} & & \Vert & & \\
0 & \longrightarrow & N & \xrightarrow{\nu'} & S & \xrightarrow{\beta'} & B & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma_N} & & \downarrow{\scriptstyle\gamma'} & & & & \\
 & & C & = & C & & & &
\end{array}$$

It follows that

$$\begin{pmatrix}\beta_M\beta'\\ \gamma_X\gamma'\end{pmatrix}(\eta',\nu') = \begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix} = \begin{pmatrix}\mu\\ \xi\end{pmatrix}f(\eta',\nu'),$$

whence $\beta_M\beta' = \mu f$ and $\gamma_X\gamma' = \xi f$. We thus have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^2 & \xrightarrow{\left(\begin{smallmatrix}f & 0\\ 0 & f\end{smallmatrix}\right)} & E^2 & \xrightarrow{\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)} & A^2 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\left(\begin{smallmatrix}\beta' & 0\\ 0 & \gamma'\end{smallmatrix}\right)} & & \downarrow{\scriptstyle\left(\begin{smallmatrix}\mu & 0\\ 0 & \xi\end{smallmatrix}\right)} & & \Vert & & \\
0 & \longrightarrow & B\oplus C & \xrightarrow{\left(\begin{smallmatrix}\beta_M & 0\\ 0 & \gamma_X\end{smallmatrix}\right)} & M\oplus X & \xrightarrow{\left(\begin{smallmatrix}\alpha_M & 0\\ 0 & \alpha_X\end{smallmatrix}\right)} & A^2 & \longrightarrow & 0
\end{array}$$

Applying the Snake Lemma, using the exact sequence

$$0 \longrightarrow N\oplus Y \xrightarrow{\left(\begin{smallmatrix}\nu' & 0\\ 0 & \eta'\end{smallmatrix}\right)} S^2 \xrightarrow{\left(\begin{smallmatrix}\beta' & 0\\ 0 & \gamma'\end{smallmatrix}\right)} B\oplus C \longrightarrow 0,$$

yields the exact sequence

$$0 \longrightarrow N\oplus Y \xrightarrow{\left(\begin{smallmatrix}\nu & 0\\ 0 & \eta\end{smallmatrix}\right)} E^2 \xrightarrow{\left(\begin{smallmatrix}\mu & 0\\ 0 & \xi\end{smallmatrix}\right)} M\oplus X \longrightarrow 0$$

as required. □

We now need to study the fibres of the map from diagrams to frames, however different frames will yield different isomorphism classes of middle term E. We therefore use the parameterising space $\mathrm{Rep}(\underline{e})$. More precisely, let us fix points in the respective parameterising spaces representing the modules A, B, C, D, M, N, X and Y. Then, given a point in $\mathrm{Rep}(\underline{e})$ representing the module E, we can identify $\mathrm{Hom}(N,E)$ for example with a subset of $\mathrm{Hom}(\underline{\dim}\,N,\underline{e})$ depending on the chosen points representing N and E.

Lemma 2.12. *There is a natural surjection*

$$\coprod_{E\in\mathrm{Rep}(\underline{e})} \mathcal{D}(A,B,C,D;E) \to \mathcal{F}(A,B,C,D)$$

with fibres of size $|\,\mathrm{GL}(\underline{e})|q^{-\langle A,D\rangle}E$.

Proof. The first statement is clear, since we are working in an hereditary category. For the second statement, let us fix a frame as well as the two homotopy squares

$$\begin{array}{ccc} D & \xrightarrow{\delta_Y} & Y \\ \big\downarrow{\scriptstyle \delta_N} & & \big\downarrow{\scriptstyle \eta'} \\ N & \xrightarrow{\nu'} & S \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{\mu'} & M \\ \big\downarrow{\scriptstyle \xi'} & & \big\downarrow{\scriptstyle \alpha_M} \\ X & \xrightarrow{\alpha_X} & A \end{array}$$

We further obtain the map $h\colon S \to R$, which we factor via its image as $h = \iota\pi$.

The previous lemma tells us that the number of completions of our frame to a 3×3 diagram is precisely the number of triples $(f, g; E)$ such that

$$\begin{array}{ccc} S & \xrightarrow{\pi} & I \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \iota} \\ E & \xrightarrow{g} & R \end{array}$$

is homotopy Cartesian. For, if $(\bar{f}, \bar{g}; E)$ is another such triple for the same E and with

$$\begin{pmatrix} \mu' \\ \xi \end{pmatrix} g = \begin{pmatrix} \mu \\ \xi \end{pmatrix} = \begin{pmatrix} \mu' \\ \xi' \end{pmatrix} \bar{g} \quad \text{and} \quad f(\nu', \eta') = (\nu, \eta) = \bar{f}(\nu', \eta'),$$

then $(f, g) = (\bar{f}, \bar{g})$ since $\binom{\mu'}{\xi'}$ is injective and (ν', η') is surjective.

To construct such a triple $(f, g; E)$, we first take an extension class $[(s, g)] \in \mathrm{Ext}^1(R, D)_E$ whose pull-back along ι is precisely the class $[(\delta', \pi)] \in \mathrm{Ext}^1(I, D)$. We then find f such that $gf = h$ and $f\delta' = s$. It is clear that all such triples arise in this way.

We define the sets

$$\begin{aligned} T &:= \{(f, g; E) : f \in \mathrm{Hom}(S, E), g \in \mathrm{Hom}(E, R), gf = h, (f\delta', g) \in \mathcal{P}^E_{RD}\} \\ &\quad\ \subset \mathrm{Hom}(\underline{\dim}\, S, \underline{e}) \times \mathrm{Hom}(\underline{e}, \underline{\dim}\, R) \times \mathrm{Rep}(\underline{e}) \\ P &:= \{(s, g; E) : (s, g) \in \mathcal{P}^E_{RD}\} \subset \mathrm{Hom}(\underline{\dim}\, D, \underline{e}) \times \mathrm{Hom}(\underline{e}, \underline{\dim}\, R) \times \mathrm{Rep}(\underline{e}). \end{aligned}$$

Consider the maps

$$T \to P \to \mathrm{Ext}^1(R, D) \to \mathrm{Ext}^1(I, D),$$

where the first map sends $(f, g; E)$ to the triple $(f\delta', g; E)$ and the last map is given by taking the pull-back along ι.

The first map has fibres of size $q^{[I,D]}$. For, if $(\bar{f}, g)$ and (f, g) map to the same point, then $f - \bar{f} = \lambda\pi$ for some $\lambda\colon I \to E$ and $g\lambda\pi = g(f - \bar{f}) = 0$. Since π is surjective, we see that $g\lambda = 0$, hence $\lambda = f\delta'\mu$ for some $\mu\colon I \to D$ (where we have used that $f\delta'$ is a kernel for g). Then $\bar{f} = f(1 - \delta'\mu\pi)$ and $1 - \delta'\mu\pi \in \mathrm{Aut}(D)$ for all $\mu \in \mathrm{Hom}(I, D)$.

The second map has fibres of size $|\,\mathrm{GL}(\underline{e})|/q^{[R,D]}$. For, as as shown in the proof of Riedtmann's Formula, Proposition 2.3, the map $\mathcal{P}^E_{RD} \to \mathrm{Ext}^1(R, D)_E$ has fibres of size $a_E/q^{[R,D]}$, whereas the number of points in $\mathrm{Rep}(\underline{e})$ isomorphic to E is $|\,\mathrm{GL}(\underline{e})|/a_E$.

Finally, the third map is surjective, since the category is hereditary, and by the long exact seqence for $\mathrm{Hom}(-, D)$ applied to

$$0 \longrightarrow I \xrightarrow{\ \iota\ } R \xrightarrow{\ \alpha'\ } A \longrightarrow 0$$

the fibres all have size $q^{[R,D]-[I,D]-\langle A,D\rangle}$.

Now, the triples $(f,g;E)\in T$ fitting into a homotopy Cartesian square as above are precisely those whose image in $\mathrm{Ext}^1(I,D)$ equals the class $[(\delta',\pi)]$. We deduce that there are exactly $|\,\mathrm{GL}(\underline{e})|q^{-\langle A,D\rangle}$ such triples. $\square$

The proof of Green's Formula, Proposition 2.7, follows easily. For, it is clear that

$$|\mathcal{F}(A,B,C,D)| = P^M_{AB}P^N_{CD}P^X_{AC}P^Y_{BD},$$
$$\sum_{A,B,C,D}|\mathcal{C}(A,B,C,D;E)| = P^E_{MN}P^E_{XY}.$$

Also, the lemmas above imply that

$$\begin{aligned} q^{-\langle A,D\rangle}\frac{P^M_{AB}P^N_{CD}P^X_{AC}P^Y_{BD}}{a_Aa_Ba_Ca_D} &= q^{-\langle A,D\rangle}\frac{|\mathcal{F}(A,B,C,D)|}{a_Aa_Ba_Ca_D}\\ &= \frac{1}{|\,\mathrm{GL}(\underline{e})|}\sum_{E\in\mathrm{Rep}(\underline{e})}\frac{|\mathcal{D}(A,B,C,D;E)|}{a_Aa_Ba_Ca_D}\\ &= \sum_E\frac{|\mathcal{D}(A,B,C,D;E)|}{a_Aa_Ba_Ca_Da_E}\\ &= \sum_E\frac{|\mathcal{C}(A,B,C,D;E)|}{a_E}. \end{aligned}$$

Note that the last two sums are over isomorphism classes of E, whereas the first sum is over points in the parameterising space $\mathrm{Rep}(\underline{e})$.

Summing over all isomorphism classes of A, B, C and D and substituting in for $|\mathcal{C}(A,B,C,D;E)|$ completes the proof.

5. Rank Two Calculations

Our first example is the n-Kronecker quiver

$$1\xrightarrow{(n,n)}2$$

The corresponding path algebra is $\Lambda=\begin{pmatrix}k & k^n\\ 0 & k\end{pmatrix}$. The projectives are given by the rows $P_1=(k,k^n)$ and $P_2=(0,k)$, and the injectives are given by the duals of the columns $I_1=D\binom{k}{0}$ and $I_2=D\binom{k^n}{k}$, where $D=\mathrm{Hom}_k(-,k)$ is the standard duality.

For a dimension vector $\underline{d}$, define

$$u_{\underline{d}} := \sum_{\substack{\underline{\dim}\,M=\underline{d}\\ M\text{ indec}}} u_M. \tag{5.1}$$

Consider first a product $u_{2^r}u_1u_{2^s}$, where $u_{i^m}:=u_{[S_i^m]}$. Clearly

$$u_1u_{2^s} = \sum_{\underline{\dim}\,M=(1,s)} u_M, \tag{5.2}$$

and since $\underline{\dim}\,M=(1,s)$, we know $M\cong N\oplus S_{2^{s-a}}$, where $\underline{\dim}\,N=(1,a)$ and N is indecomposable. Moreover, since $\mathrm{Hom}(N,S_2)=0$, we can use Riedtmann's

Formula to deduce that $F^{S_2^t\oplus N}_{S_2^t N}=1$. Thus $u_{2^t}u_N=u_{[S_2^t\oplus N]}$, and so

$$u_{2^r}u_1u_{2^s}=\sum_{a=0}^{s}u_{2^r}u_{2^{s-a}}u_{(1,a)}=\sum_{a=0}^{s}\begin{bmatrix}r+s-a\\r\end{bmatrix}_+u_{2^{r+s-a}}u_{(1,a)}. \tag{5.3}$$

We have used here that $\mathrm{End}(S_i)=k$ and $\mathrm{Ext}^1(S_i,S_i)=0$, so that the subcategory $\mathrm{add}(S_i)$ is isomorphic to the category of k-vector spaces.

We now note that $u_{(1,m)}=0$ for $m>n$, and $u_{(1,n)}=u_{[P_1]}$. There are two ways of seeing this. Let N be a module of dimension vector $(1,m)$. First, we may consider the corresponding representation $\mu\colon k\otimes_k k^n\to k^m$. If $m>n$, then $\dim\mathrm{Im}(\mu)\le n<m$. Hence we can find a non-zero complement to the image, or in other words S_2 is a direct summand. If $m=n$, then the representation is indecomposable if and only if μ is an isomorphism, or in other words $N\cong P_1$. The second way is more categorical. There is an epimorphism $N\to S_1$, hence a morphism $f\colon P_1\to N$. The cokernel of $\mathrm{Im}(f)$ has dimension vector $(0,t)$ for some t, hence is isomorphic to S_2^t, which is projective. Thus N is indecomposable if and only if $N\cong\mathrm{Im}(f)$. It is now clear that $\underline{\dim}\,N\le\underline{\dim}\,P_1=(1,n)$, and we have equality if and only if f is an isomorphism.

Coming back to our products $u_{2^r}u_1u_{2^s}$ we see that for $r+s=n+1$ there are $n+2$ possible products, but only $n+1$ possible summands $u_{2^{n+1-a}}u_{(1,a)}$ for $0\le a\le n$. Hence we have a relation of the form

$$\sum_{r+s=n+1}\lambda_r u_{2^r}u_1u_{2^s}=\sum_{a=0}^{n}\Big(\sum_{r=0}^{n+1-a}\lambda_r\begin{bmatrix}n+1-a\\r\end{bmatrix}_+\Big)u_{2^{n+1-a}}u_{(1,a)}=0. \tag{5.4}$$

We need to determine the coefficients λ_r such that $\sum_{r=0}^{n+1-a}\lambda_r\left[\begin{smallmatrix}n+1-a\\r\end{smallmatrix}\right]_+=0$ for all $0\le a\le n$.

Consider first the corresponding result for the ordinary binomial coefficients. We have

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}=0\quad\text{for all } m. \tag{5.5}$$

The proof is easy, using the binomial formula

$$(1+x)^m=\sum_{r=0}^{m}\binom{m}{r}x^r \tag{5.6}$$

and setting $x=-1$.

We emulate this proof for the quantum binomial coefficients, using the formula

$$(1+x)(1+qx)\cdots(1+q^{m-1}x)=\sum_{r=0}^{m}q^{\binom{r}{2}}\begin{bmatrix}m\\r\end{bmatrix}_+x^r. \tag{5.7}$$

The proof can be done by induction on m. If we multiply both sides by $1+q^mx$, then the result follows from the identity

$$q^{\binom{r}{2}}\begin{bmatrix}m\\r\end{bmatrix}_++q^{m+\binom{r-1}{2}}\begin{bmatrix}m\\r-1\end{bmatrix}_+=q^{\binom{r}{2}}\begin{bmatrix}m+1\\r\end{bmatrix}_+.$$

Setting $x=-1$ yields the identity

$$\sum_{r=0}^{m}(-1)^rq^{\binom{r}{2}}\begin{bmatrix}m\\r\end{bmatrix}_+=0\quad\text{for all } m. \tag{5.8}$$

Our coefficients must therefore be $\lambda_r = (-1)^r q^{\binom{r}{2}}$. (Setting $x = -q^{-t}$ gives zero only when $m \geq t+1$, but we need to obtain zero for all $m \geq 1$, so the solution is unique up to scalars.)

We thus have the relation

$$\sum_{r+s=n+1} (-1)^r q^{\binom{r}{2}} u_{2^r} u_1 u_{2^s} = 0. \tag{5.9}$$

Similarly,

$$u_{1^r} u_2 u_{1^s} = \sum_{a=0}^{r} \begin{bmatrix} r+s-a \\ s \end{bmatrix}_+ u_{(a,1)} u_{1^{r+s-a}}. \tag{5.10}$$

Note that, since S_1 is injective, multiplying on the right by copies of S_1 yields direct summands, whereas it was multiplication on the left for the projective S_2.

Again, $u_{(m,1)} = 0$ for $m > n$, whereas $u_{(n,1)} = u_{[I_2]}$. Hence we obtain the relation

$$\sum_{r+s=n+1} (-1)^s q^{\binom{s}{2}} u_{1^r} u_2 u_{1^s} = 0. \tag{5.11}$$

Our second example is the cyclic quiver

$$1 \rightleftarrows 2$$

In this case the category $\operatorname{mod}\Lambda$ is uniserial, without any (non-zero) projective or injective objects. We simplify the notation by writing, for example $\substack{2\\1\;1^r}$ for the module $S_2(2) \oplus S_1^r$, where $S_2(2)$ has simple top S_2 and Loewy length 2. We then have the identities

$$\begin{aligned} u_2 u_{1^s} &= u_{\substack{2\\1\;1^{s-1}}} + u_{1^s 2} \\ u_{1^r} u_2 u_{1^s} &= \begin{bmatrix} r+s-2 \\ r-1 \end{bmatrix}_+ u_{\substack{1\\2\\1\;1^{r+s-2}}} + \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_+ u_{\substack{2\\1\;1^{r+s-1}}} \\ &\quad + \begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix}_+ u_{\substack{1\\2\;1^{r+s-1}}} + \begin{bmatrix} r+s \\ r \end{bmatrix}_+ u_{1^{r+s}2}. \end{aligned} \tag{5.12}$$

For fixed $r+s$, there are at most four possible summands, hence there is definitely a relation when $r+s=4$. We can do better, though, since there is already a relation when $r+s=3$. In detail, we have the products

$$\begin{aligned} u_{1^3} u_2 &= u_{\substack{1\\2\;1^2}} + u_{1^3 2} \\ u_{1^2} u_2 u_1 &= u_{\substack{1\\2\\1\;1}} + [2]_+ u_{\substack{1\\2\;1^2}} + u_{\substack{2\\1\;1^2}} + [3]_+ u_{1^3 2} \\ u_1 u_2 u_{1^2} &= u_{\substack{1\\2\\1\;1}} + u_{\substack{1\\2\;1^2}} + [2]_+ u_{\substack{2\\1\;1^2}} + [3]_+ u_{1^3 2} \\ u_2 u_{1^3} &= u_{\substack{2\\1\;1^2}} + u_{1^3 2}. \end{aligned} \tag{5.13}$$

We see that

$$u_{1^2} u_2 u_1 - u_1 u_2 u_{1^2} = q\big(u_{\substack{1\\2\;1^2}} - u_{\substack{2\\1\;1^2}}\big) = q\big(u_{1^3} u_2 - u_2 u_{1^3}\big), \tag{5.14}$$

whence (by symmetry) the relations

$$\begin{aligned} q u_{1^3} u_2 - u_{1^2} u_2 u_1 + u_1 u_2 u_{1^2} - q u_2 u_{1^3} &= 0 \\ q u_1 u_{2^3} - u_2 u_1 u_{2^2} + u_{2^2} u_1 u_2 - q u_{2^3} u_1 &= 0. \end{aligned} \tag{5.15}$$

As expected, the relations depend upon the chosen orientation, as of course do the categories $\operatorname{mod}\Lambda$.

Exercises 6.

(1) Calculate the relations for the species $\Lambda = \begin{pmatrix} k & K \\ 0 & K \end{pmatrix}$ where $[K:k]=n$, corresponding to the valued graph

$$1 \xrightarrow{(n,1)} 2$$

where we note that these must be homogeneous of degrees $(n+1,1)$ and $(1,2)$.

(2) What about an arbitrary rank two species?

6. Twisting the Bialgebra Structure

Rather surprisingly, Ringel [Hall algebras revisited] showed how to remove the dependence of the Ringel-Hall algebra on the chosen orientation by twisting the multiplication with respect to the Euler form.

Let $v \in \mathbb{C}$ be a square-root of q, and denote by $\mathbb{Q}_v \subset \mathbb{C}$ the subfield generated by $v^{\pm 1}$. We define a new multiplication on $\mathbb{Q}_v \otimes_{\mathbb{Z}} \mathcal{H}(\mathcal{A})$ via

$$u_M * u_N := \sum_X v^{\langle M,N\rangle} F^X_{MN} u_X = v^{\langle M,N\rangle} u_M u_N. \tag{6.1}$$

It is quickly checked that this multiplication is again associative (since the Euler form is bilinear), has the same unit u_0 and respects the grading.

We consider the relations worked out in the previous section. For the n-Kronecker quiver

$$1 \xrightarrow{(n,n)} 2$$

the Euler form $\langle -,-\rangle$ corresponds to the matrix $R = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$. We begin by rewriting u_{i^r} as the divided power $\frac{1}{[r]_+!}u_i^r$. The relations from the previous section thus read

$$\begin{aligned} \sum_{r+s=n+1} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} n+1 \\ s \end{bmatrix}_+ u_1^r u_2 u_1^s = 0 \\ \sum_{r+s=n+1} (-1)^r q^{\binom{r}{2}} \begin{bmatrix} n+1 \\ r \end{bmatrix}_+ u_2^r u_1 u_2^s = 0. \end{aligned} \tag{6.2}$$

Now,

$$u_1^{*r} * u_2 * u_1^{*s} = v^{\binom{r}{2}+\binom{s}{2}} u_1^r * u_2 * u_1^s = v^{\binom{r}{2}+\binom{s}{2}-rn+rs} u_1^r u_2 u_1^s. \tag{6.3}$$

We also have the following relations between the two types of quantum number $[m]_+ = \frac{q^m-1}{q-1}$ and $[m] = \frac{v^m - v^{-m}}{v-v^{-1}}$:

$$\begin{aligned} [m]_+ &= v^{m-1}[m], \\ [m]_+! &= v^{\binom{m}{2}}[m]! \end{aligned} \quad \text{and} \quad \begin{bmatrix} m \\ a \end{bmatrix}_+ = v^{\binom{m}{2}-\binom{a}{2}-\binom{m-a}{2}} \begin{bmatrix} m \\ a \end{bmatrix} = v^{a(m-a)} \begin{bmatrix} m \\ a \end{bmatrix}. \tag{6.4}$$

Thus

$$\begin{aligned}
&q^{\binom{s}{2}} \begin{bmatrix} n+1 \\ s \end{bmatrix}_+ u_1^r u_2 u_1^s \\
&\quad = v^{2\binom{s}{2}} \cdot v^{\binom{n+1}{s}-\binom{s}{2}-\binom{r}{2}} \begin{bmatrix} n+1 \\ s \end{bmatrix} \cdot v^{-\binom{r}{2}-\binom{s}{2}+r(n-s)} u_1^{*r} * u_2 * u_1^{*s} \\
&\quad = v^{\binom{n+1}{2}} \begin{bmatrix} n+1 \\ s \end{bmatrix} u_1^{*r} * u_2 * u_1^{*s} = v^{\binom{n+1}{2}} [n+1]! u_1^{(*r)} * u_2 * u_1^{(*s)},
\end{aligned} \tag{6.5}$$

where we have again used the divided powers

$$u_1^{(*r)} := \frac{1}{[r]!} u_1^{*r}. \tag{6.6}$$

Therefore we obtain the following relations

$$\begin{aligned}
\sum_{r+s=n+1} (-1)^s u_1^{(*r)} * u_2 * u_1^{(*s)} &= 0 \\
\sum_{r+s=n+1} (-1)^r u_2^{(*r)} * u_1 * u_2^{(*s)} &= 0.
\end{aligned} \tag{6.7}$$

Next we consider the cyclic quiver

$$1 \rightleftarrows 2$$

In this case, the Euler form is given via the matrix $R = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. We apply the same sequence of calculations, so we have the original relation

$$\begin{aligned}
& qu_{1^3}u_2 - u_{1^2}u_2u_1 + u_1u_2u_{1^2} - qu_2u_{1^3} = 0, \\
\rightsquigarrow \quad & qu_1^3u_2 - [3]_+u_1^2u_2u_1 + [3]_+u_1u_2u_1^2 - qu_2u_1^3 = 0, \\
\rightsquigarrow \quad & v^2u_1^{*3} * u_2 - v^2[3]u_1^{*2} * u_2 * u_1 + v^2[3]u_1 * u_2 * u_1^{*2} - v^2u_2 * u_1^{*3} = 0.
\end{aligned}$$

We can now divide by $v^2[3]!$ and take divided powers to obtain the relations

$$\begin{aligned}
\sum_{r+s=3} (-1)^r u_1^{(*r)} * u_2 * u_1^{(*s)} &= 0 \\
\sum_{r+s=3} (-1)^s u_2^{(*r)} * u_1 * u_2^{(*s)} &= 0.
\end{aligned} \tag{6.8}$$

Thus the relations for this twisted multiplication are independent of the orientation.

The natural question is whether we still have a twisted bialgebra, and if so, what is the twist required on the tensor product.

Define

$$\Delta^*(u_X) := \sum_{M,N} v^{\langle M,N\rangle} \frac{P_{MN}^X}{a_X} u_M \otimes u_N. \tag{6.9}$$

This is again coassociative with the same counit and respects the grading.

Consider $\Delta^*(u_M * u_N)$. We have, using Green's Formula,

$$\begin{aligned}\Delta^*(u_m * u_N) &= \sum_E v^{\langle M,N\rangle} \frac{P_{MN}^E}{a_M a_N} \Delta^*(u_E) \\ &= \sum_{X,Y,E} v^{\langle M,N\rangle+\langle X,Y\rangle} \frac{P_{MN}^E P_{XY}^E}{a_M a_N a_E} u_X \otimes u_Y \\ &= \sum_{\substack{A,B,C,D \\ X,Y}} v^{\langle M,N\rangle+\langle X,Y\rangle-2\langle A,D\rangle} \frac{P_{AB}^M P_{CD}^N P_{AC}^X P_{BD}^Y}{a_A a_B a_C a_D a_M a_N} u_X \otimes u_Y.\end{aligned}$$

Here we have used the identity $q^{-\langle A,D\rangle} = v^{-2\langle A,D\rangle}$. Next substitute in for

$$\sum_X \frac{P_{AC}^X}{a_A a_C} u_X = v^{-\langle A,C\rangle} u_A * u_C \quad \text{and} \quad \sum_Y \frac{P_{BD}^Y}{a_B a_D} u_Y = v^{-\langle B,D\rangle} u_B * u_D,$$

as well as for

$$\langle M,N\rangle + \langle X,Y\rangle = \langle A+B, C+D\rangle + \langle A+C, B+D\rangle.$$

This yields

$$\Delta^*(u_M * u_N) = \sum_{A,B,C,D} v^{\langle A,B\rangle+\langle C,D\rangle+(B,C)} \frac{P_{AB}^M P_{CD}^N}{a_M a_N} (u_A * u_C) \otimes (u_B * u_D), \tag{6.10}$$

where we have used the symmetric bilinear form

$$(B,C) := \langle B,C\rangle + \langle C,B\rangle. \tag{6.11}$$

On the other hand, we wish to compare this to the following, with respect to some twisted multiplication on the tensor product:

$$\Delta^*(u_M) * \Delta^*(u_N) = \sum_{A,B,C,D} v^{\langle A,B\rangle+\langle C,D\rangle} \frac{P_{AB}^M P_{CD}^N}{a_M a_N} (u_A \otimes u_B) * (u_C \otimes u_D). \tag{6.12}$$

We see that we again have a twisted bialgebra, where we have to use the following twisted multiplication on the tensor product

$$(u_A \otimes u_B) * (u_C \otimes u_D) := v^{(B,C)} (u_A * u_C) \otimes (u_B * u_D). \tag{6.13}$$

Since we now have a symmetric bilinear form, we can remove this twist on the tensor product, and hence obtain a true bialgebra, by adjoining a copy of the group algebra of the Grothendieck group $K = K(\mathcal{A})$ of $\mathcal{A}$:

$$\mathbb{Z}_v[K(\mathcal{A})] = \mathbb{Z}_v[\{K_{\underline{d}} : \underline{d} \in K(\mathcal{A})\}], \quad \text{where} \quad K_{\underline{d}} K_{\underline{e}} = K_{\underline{d}+\underline{e}}. \tag{6.14}$$

This we do in the next section. We also write $K_M = K_{\underline{\dim} M}$ for a module M.

7. The extended, twisted Ringel-Hall algebra

We now remove the $*$ from the notation and define the extended, twisted Ringel-Hall algebra $\mathrm{H} = \mathrm{H}(\mathcal{A})$ as follows.

DEFINITION 2.13. The (extended, twisted) Ringel-Hall algebra $\mathrm{H}(\mathcal{A})$ is a $\mathbb{Q}_v$-algebra which contains the group algebra $\mathbb{Z}_v[K(\mathcal{A})]$ and which is free as a right

$\mathbb{Z}_v[K(\mathcal{A})]$-module, with basis u_M indexed by the isomorphism classes of objects in $\mathcal{A}$. The multiplication and comultiplication are given via

$$\begin{aligned} u_M u_N &:= \sum_X v^{\langle M,N\rangle} \frac{P^X_{MN}}{a_M a_N} u_X, \quad & \Delta(u_X) &:= \sum_{M,N} v^{\langle M,N\rangle} \frac{P^X_{MN}}{a_X} u_M K_N \otimes u_N, \\ K_{\underline{d}} u_M &:= v^{(\underline{d},M)} u_M K_{\underline{d}}, & \Delta(K_{\underline{d}}) &:= K_{\underline{d}} \otimes K_{\underline{d}}, \end{aligned} \tag{7.1}$$

with unit $1 = u_0 = K_0$ and counit $\epsilon(u_M K_{\underline{d}}) = \delta_{M,0}$.

THEOREM 2.14. *The (extended, twisted) Ringel-Hall algebra* H *is a graded bialgebra.*

PROOF. We observe that

$$\begin{aligned} (u_A K_B \otimes u_B)(u_C K_D \otimes u_D) &= (u_A K_B u_C K_D) \otimes (u_B u_D) \\ &= v^{(B,C)} (u_A u_C K_{B+D}) \otimes (u_B u_D). \end{aligned}$$

The theorem now follows from all our previous considerations. □

In fact, we had a choice in defining the comultiplication, since all we needed was to obtain the twist $v^{(B,C)}$ in the tensor product as above. This we can do in two ways:

$$\begin{aligned} (u_A K_B \otimes u_B)(u_C K_D \otimes u_D) &= v^{(B,C)} (u_A u_C K_{B+D}) \otimes (u_B u_D) \\ (u_A \otimes K_{-A} u_B)(u_C \otimes K_{-C} u_D &= v^{(B,C)} (u_A u_C \otimes K_{-(A+C)} u_B u_D. \end{aligned}$$

Thus we could also have taken the comultiplication

$$\Delta'(u_X) := \sum_{M,N} v^{\langle M,N\rangle} \frac{P^X_{MN}}{a_X} u_M \otimes K_{-M} u_N, \quad \Delta'(K_{\underline{d}}) := K_{\underline{d}} \otimes K_{\underline{d}}. \tag{7.2}$$

The corresponding bialgebra will be denoted by H$'$. Both versions can be found in the literature, and both are needed, since they give the positive and negative parts of the corresponding quantum group.

8. Bialgebras and Hopf Algebras

We now review some general theory about bialgebras and Hopf algebras, before applying this to the Ringel-Hall algebra. Our main reference is Kassel.

Let $H = (H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra over k, where μ and η denote the multiplication and unit, and Δ and ϵ denote the comultiplication and conunit. Let $\tau \colon H \otimes H \to H \otimes H$ be the usual flip, sending $x \otimes y$ to $y \otimes x$.

Given H, we can define several other related bialgebras. These are

$$\begin{aligned} H^{\text{op}} &:= (H, \mu\tau, \eta, \Delta, \epsilon) && \text{the opposite bialgebra,} \\ H^{\text{cop}} &:= (H, \mu, \eta, \tau\Delta, \epsilon) && \text{the co-opposite bialgebra,} \\ H^{\text{op-cop}} &:= (H, \mu\tau, \eta, \tau\Delta, \epsilon) && \text{the opposite and co-opposite bialgebra.} \end{aligned} \tag{8.1}$$

When $H = \bigoplus_{n\geq 0} H_n$ is a graded bialgebra with finite dimensional graded pieces H_n, we can form the (graded) dual $H^* := \bigoplus_{n\geq 0} H_n^*$. This is again a bialgebra.

To see this, we first record a useful lemma.

LEMMA 2.15. *Let U, U', V and V' be k-vector spaces. Then there is a natural map*

$$\operatorname{Hom}(U,U')\otimes\operatorname{Hom}(V,V')\to\operatorname{Hom}(V\otimes U,U'\otimes V'),\quad (f\otimes g)\mapsto\big(v\otimes u\mapsto f(u)\otimes g(v)\big).$$

This is an isomorphism provided one of the pairs (U,U'), (V,V') or (U,V) consists of two finite dimensional vector spaces.

In particular we recover the natural maps

$$U^*\otimes V'\to\operatorname{Hom}(U,V'),\quad U'\otimes V^*\to\operatorname{Hom}(V,U'),\quad U^*\otimes V^*\to(V\otimes U)^*. \tag{8.2}$$

Now, for H as above, we have the natural maps

$$H^*\otimes H^*\to(H\otimes H)^*\xrightarrow{\Delta^*}H^*,\qquad k\xrightarrow{\epsilon^*}H^*,\qquad H^*\xrightarrow{\eta^*}k. \tag{8.3}$$

In particular, H^* is an algebra. Since each H_i is finite dimensional, we have isomorphisms $(H_i\otimes H_j)^*\cong H_i^*\otimes H_j^*$. These yield

$$\mu_n\colon\bigoplus_{i+j=n}H_i\otimes H_j\to H_n\quad\text{and}\quad\mu_n^*\colon H_n^*\to\bigoplus_{i+j=n}H_i^*\otimes H_j^*. \tag{8.4}$$

Thus

$$\bigoplus_n\mu_n^*\colon H^*\to H^*\otimes H^* \tag{8.5}$$

defines a comultiplication on H^*, and H^* is naturally a bialgebra.

An antipode for H is an endomorphism S such that

$$\sum S(x_1)x_2=\eta\epsilon(x)=\sum x_1S(x_2),\quad\text{for all } x,\text{ where }\Delta(x)=\sum x_1\otimes x_2. \tag{8.6}$$

In other words, S must satisfy the relations

$$\mu(S\otimes\mathrm{id})\Delta=\eta\epsilon=\mu(\mathrm{id}\otimes S)\Delta. \tag{8.7}$$

Now, for H a bialgebra, we can equip $\operatorname{End}_k(H)$ with a convolution product, defined via

$$f*g:=\mu(f\otimes g)\Delta,\quad\text{so that } (f*g)(x)=\sum f(x_1)g(x_2). \tag{8.8}$$

LEMMA 2.16. *The convolution product gives $\operatorname{End}_k(H)$ the structure of an associative algebra, with unit $\eta\epsilon$.*

The antipode is therefore an inverse of the identity with respect to the convolution product, provided this exists. As such, it is necessarily unique.

LEMMA 2.17.
(1) *The map S is an antipode for H if and only if it is an antipode for $H^{\text{op-cop}}$. In this case, $S\colon H\to H^{\text{op-cop}}$ is a morphism of bialgebras.*
(2) *If H is a graded bialgebra, then S is an antipode for H if and only if S^* is an antipode for H^*.*
(3) *If S is an antipode for H, then H^{op} has an antipode if and only if S is invertible, in which case the antipode equals S^{-1}.*

Note that $S\colon H\to H^{\text{op-cop}}$ is a morphism of bialgebras if and only if the following formulae hold:

$$\begin{aligned}\mu(S\otimes S)&=S\mu\tau,\quad S\eta=\eta,\\(S\otimes S)\Delta&=\tau\Delta S,\quad \epsilon S=\epsilon.\end{aligned} \tag{8.9}$$

In particular, we have that

$$S(xy) = S(y)S(x) \quad \text{and} \quad S(1) = 1. \tag{8.10}$$

Proof. We prove only the third assertion. Suppose that S' is an antipode for H^{op}. Then $\sum S'(x_2)x_1 = \eta\epsilon(x)$ for all x. Consider now $S * SS'$. We have

$$(S * SS')(x) = \sum S(x_1)SS'(x_2) = S\big(\sum S'(x_2)x_1\big) = S\big(\eta\epsilon(x)\big) = \eta\epsilon(x).$$

In particular, SS' is an inverse for S with respect to the convolution product, hence $SS' = \mathrm{id}$.

Conversely, if S is invertible, then we can apply S to both $\sum x_2 S^{-1}(x_1)$ and $\sum S^{-1}(x_2)x_1$ to show that S^{-1} is an antipode for H^{op}. □

For bialgebras A and B over k, let $(-,-)\colon A \times B \to k$ be a k-bilinear map. We also have the induced map

$$(-,-)\colon (A \otimes A) \times (B \otimes B) \to k, \quad (a \otimes a', b \otimes b') := (a,b)(a',b'). \tag{8.11}$$

The map $(-,-)$ is called a bialgebra pairing if the following properties are satisfied:

(1) $(aa', b) = (a \otimes a', \Delta(b))$ and $(a, bb') = (\Delta(a), b \otimes b')$;
(2) $(a, 1) = \epsilon(a)$ and $(1, b) = \epsilon(b)$.

If A and B are both Hopf algebras, then $(-,-)$ is called a Hopf pairing provided we have the additional property

(3) $(S(a), b) = (a, S(b))$.

There is a natural pairing $H^* \otimes H \to k$ given by evaluation.

Exercise 7. Prove the following isomorphism of bialgebras:

$$\mathrm{H}'(\mathcal{A}^{\mathrm{op}}) \cong \mathrm{H}(\mathcal{A})^{\mathrm{op-cop}}.$$

9. The Antipode for the Ringel-Hall Algebra

Define an endomorphism S of H via $S(K_{\underline{d}}) := K_{-\underline{d}}$ and, for $M \neq 0$,

$$S(u_M K_{-M}) := \sum_{r \geq 1} \sum_{\substack{X_1,\dots,X_r \\ \text{non-zero}}} (-1)^r v^{2\sum_{i<j}\langle X_i, X_j\rangle} \frac{a_{X_1}\cdots a_{X_r}}{a_M} F^M_{X_1\cdots X_r} \sum_N F^N_{X_1\cdots X_r} u_N. \tag{9.1}$$

Note that for $r = 1$ we obtain $-u_M$. In general we set

$$S(u_M K_{\underline{d}}) := K_{-(M+\underline{d})} S(u_M K_{-M}). \tag{9.2}$$

Theorem 2.18 (Xiao). *The map S is an antipode for* H.

Proof. We wish to show that $S * \mathrm{id} = \eta\epsilon$, the result for $\mathrm{id} * S$ being dual. This is clear for $K_{\underline{d}}$, so consider $u_M K_{-M}$. We have

$$\begin{aligned}\Delta(u_M K_{-M}) &= \sum_{A,B} v^{\langle A,B\rangle} \frac{a_A a_B}{a_M} F^M_{AB} u_A K_{-A} \otimes u_B K_{-M} \\ &= u_M K_{-M} \otimes K_{-M} + 1 \otimes u_M K_{-M} \\ &\quad + \sum_{\substack{A,B \\ \text{non-zero}}} v^{\langle A,B\rangle} \frac{a_A a_B}{a_M} F^M_{AB} u_A K_{-A} \otimes u_B K_{-M}.\end{aligned}$$

Applying $\mu(S\otimes\mathrm{id})$ thus yields

$$(S*\mathrm{id})(u_MK_{-M}) = S(u_MK_{-M})K_{-M} + u_MK_{-M} + \sum_{\substack{A,B\\ \text{non-zero}}} v^{\langle A,B\rangle}\frac{a_Aa_B}{a_M}F^M_{AB}S(u_AK_{-A})u_BK_{-M}.$$

Substituting in for $S(u_AK_{-A})$ shows that $\big((S*\mathrm{id})(u_MK_{-M})\big)K_M$ equals

$$S(u_MK_{-M}) + u_M + \sum_{r\geq 1}\sum_{\substack{X_1,\dots,X_r,B\\ \text{non-zero}}} v^{\langle X_1+\cdots+X_r,B\rangle+2\sum_{i<j}\langle X_i,X_j\rangle}\frac{a_{X_1}\cdots a_{X_r}a_B}{a_M} \times\sum_A F^A_{X_1\cdots X_r}F^M_{AB}\sum_L F^L_{X_1\cdots X_r}u_Lu_B.$$

We now observe that

$$\sum_A F^A_{X_1\cdots X_r}F^M_{AB} = F^M_{X_1\cdots X_rB}$$

and that

$$\sum_L F^L_{X_1\cdots X_r}u_Lu_B = v^{\langle X_1+\cdots+X_r,B\rangle}\sum_{L,N}F^L_{X_1\cdots X_r}F^N_{LB}u_N = v^{\langle X_1+\cdots+X_r,B\rangle}\sum_N F^N_{X_1\cdots X_rB}u_N.$$

Substituting back in, setting $X_{r+1}:=B$, and then replacing r by $r-1$ we obtain

$$\begin{aligned}(S*\mathrm{id})&(u_MK_{-M})K_M\\ &= S(u_MK_{-M}) + u_M\\ &\quad -\sum_{r\geq 2}\sum_{\substack{X_1,\dots,X_r\\ \text{non-zero}}} v^{2\sum_{i<j}\langle X_i,X_j\rangle}\frac{a_{X_1}\cdots a_{X_r}}{a_M}F^M_{X_1\cdots X_r}\sum_N F^N_{X_1\cdots X_r}u_N\\ &= 0.\end{aligned}$$

The general case for $u_MK_{\underline{d}}$ follows immediately. □

COROLLARY 2.19. *The antipode is invertible, with inverse given by* $S'(K_{\underline{d}}) := K_{-\underline{d}}$ *and, for* $M\neq 0$,

$$S'(K_{-M}u_M) := \sum_{r\geq 1}\sum_{\substack{X_1,\dots,X_r\\ \text{non-zero}}}(-1)^r v^{2\sum_{i<j}\langle X_i,X_j\rangle}\frac{a_{X_1}\cdots a_{X_r}}{a_M}F^M_{X_1\cdots X_r}\sum_N F^N_{X_r\cdots X_1}u_N.$$

In general, we set

$$S'(K_{\underline{d}}u_M) := S'(K_{-M}u_M)K_{-(M+\underline{d})}.$$

PROOF. Essentially the same proof shows that S' is an antipode for the opposite bialgebra H^{op}. The result then follows from standard results for Hopf algebras (as in the previous section). □

10. Green's Hopf Pairing

In this section we define Green's Hopf pairing, and hence show that the Ringel-Hall algebra H is a self-dual Hopf algebra.

PROPOSITION 2.20. *The symmetric bilinear form on* H × H *given via*

$$\{u_M K_{\underline{d}}\,,\, u_N K_{\underline{e}}\} := \delta_{MN} \frac{v^{(\underline{d},\underline{e})}}{a_M}$$

is a Hopf pairing. Hence H *is a self-dual Hopf algebra.*

PROOF. We need to check the three axioms of a Hopf pairing. Since u_M and u_N are orthogonal for $M \not\cong N$, we have

$$\begin{aligned}\{u_M K_{\underline{d}}\,,\, u_N K_{\underline{e}} u_P K_{\underline{f}}\} &= \{u_M K_{\underline{d}}\,,\, v^{\langle N,P\rangle + (P,\underline{e})} F^M_{NP} u_M K_{\underline{e}+\underline{f}}\} \\ &= v^{\langle N,P\rangle + (P,\underline{e}) + (\underline{d},\underline{e}+\underline{f})} F^M_{NP}/a_M.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\{\Delta(u_M K_{\underline{d}})\,,\, u_N K_{\underline{e}} \otimes u_P K_{\underline{f}}\} \\ &\qquad = \{v^{\langle N,P\rangle} F^M_{NP} \tfrac{a_N a_P}{a_M} u_N K_{P+\underline{d}} \otimes u_P K_{\underline{d}}\,,\, u_N K_{\underline{e}} \otimes u_P K_{\underline{f}}\} \\ &\qquad = v^{\langle N,P\rangle + (P+\underline{d},\underline{e}) + (\underline{d},\underline{f})} F^M_{NP}/a_M.\end{aligned}$$

Hence the multiplication and comultiplication are adjoint to one another.

Clearly

$$\{u_M K_{\underline{d}}, 1\} = \delta_{M0} = \epsilon(u_M K_{\underline{d}}).$$

Finally, consider the antipode. We have

$$\begin{aligned}&\{S(u_M K_{\underline{d}})\,,\, u_N K_{\underline{e}}\} = \{S(K_{M+\underline{d}}) S(u_M K_{-M})\,,\, u_N K_{\underline{e}}\} \\ &\quad = v^{-(M+\underline{d},N+\underline{e})} \sum_{r\geq 1} \sum_{\substack{X_1,\ldots,X_r \\ \text{non-zero}}} (-1)^r v^{2\sum_{i<j}\langle X_i,X_j\rangle} F^M_{X_1\cdots X_r} F^N_{X_1\cdots X_r} \frac{a_{X_1}\cdots a_{X_r}}{a_M a_N}.\end{aligned}$$

Since this is symmetric in $(M,\underline{d})$ and $(N,\underline{e})$, we see that

$$\{S(u_M K_{\underline{d}})\,,\, u_N K_{\underline{e}}\} = \{u_M K_{\underline{d}}\,,\, S(u_N K_{\underline{e}})\}$$

as required. □

11. Primitive Generators for Ringel-Hall Algebras

We finish this chapter with a result of Sevenhant and Van den Bergh showing that the Ringel-Hall algebra is generated by primitive elements.

Define H^+ to be the subalgebra generated by the u_M. Then Green's Hopf pairing restricts to

$$\{u_M, u_N\} := \delta_{MN}/a_M.$$

Since a_M is a positive integer, we see that the form on H^+ is positive definite.

Now, for each α, H^+_α is finite dimensional and contains the subspace

$$\sum_{\beta+\gamma=\alpha;\,\beta,\gamma\neq 0} \mathrm{H}^+_\beta\, \mathrm{H}^+_\gamma\,.$$

Hence we can find an orthogonal basis for the orthogonal complement

$$\Big(\sum_{\beta+\gamma=\alpha;\beta,\gamma\neq 0} \mathrm{H}^+_\beta\, \mathrm{H}^+_\gamma\Big)^\perp.$$

Let $\{\theta_i\}_{i\in I}$ be the union of all of these bases for all $\alpha > 0$.

We note that if $\alpha = e_i$ for some $i \in Q_0$, then $\mathrm{H}^+_\alpha = \mathbb{Q}_v u_{S_i}$ and so without loss of generality we may assume that $\theta_i = u_{S_i}$. In this way we identify $Q_0 \subset I$.

LEMMA 2.21. *For each $i \in I$ we have*

$$\Delta(\theta_i) = \theta_i \otimes 1 + K_i \otimes \theta_i,$$

where $K_i := K_{\underline{\dim}\,\theta_i}$.

PROOF. Extend $\{\theta_i\}$ to a homogeneous orthogonal basis $\{f_r\}$ for H^+. We may assume that $f_0 = 1$. Set $\xi_r := \{f_r, f_r\} \in \mathbb{Q}_v^+$, a positive real number.

Write $\Delta(\theta_i) = \sum_{r,s} c_{rs} f_r K_{\underline{\dim}\,f_s} \otimes f_s$. Then

$$\{\theta_i, f_r f_s\} = \{\Delta(\theta_i), f_r \otimes f_s\} = \xi_r \xi_s c_{rs}.$$

On the other hand, if both f_r and f_s are different from 1, then this is zero by definition of θ_i (it is orthogonal to all proper products). Hence $c_{rs} = 0$. Also,

$$\{\theta_i, f_r\} = \xi_r c_{r0} = \xi_r c_{0r}$$

so that $c_{r0} = c_{0r} = 0$ if $f_r \neq \theta_i$, whereas $c_{r0} = c_{0r} = 1$ if $f_r = \theta_i$. □

We can define a symmetric bilinear form on the lattice $\mathbb{Z}I$ via

$$(i, j) := (\underline{\dim}\,\theta_i, \underline{\dim}\,\theta_j). \tag{11.1}$$

This clearly extends the bilinear form on $\mathbb{Z}Q_0 \subset \mathbb{Z}I$.

PROPOSITION 2.22. *This bilinear form satisfies the following properties.*

(1) *$(i, j) \leq 0$ for all $i \neq j$, and $(i, j) = 0$ implies $\theta_i\theta_j = \theta_j\theta_i$;*
(2) *$(i, i) \in 2\mathbb{Z}$, and $i \notin Q_0$ implies $(i, i) \leq 0$;*
(3) *Set $s_i := 1$ for $i \notin Q_0$. Then $\frac{1}{s_i}(i, j) \in \mathbb{Z}$ for all $i, j \in I$.*

In particular, $C = D^{-1}B$ is a symmetrisable Borcherds matrix, where $D = \mathrm{diag}(s_i)$ and $B = ((i, j))$.

PROOF. Consider

$$\begin{aligned}\Delta(\theta_i\theta_j) &= (\theta_i \otimes 1 + K_i \otimes \theta_i)(\theta_j \otimes 1 + K_j \otimes \theta_j)\\ &= \theta_i\theta_j \otimes 1 + \theta_i K_j \otimes \theta_j + v^{(i,j)}\theta_j K_i \otimes \theta_i + K_{i+j} \otimes \theta_i\theta_j.\end{aligned}$$

Using the adjointness of the multiplication and comultiplication, it follows that

$$\{\theta_i\theta_j, \theta_i\theta_j\} = \xi_i\xi_j \quad \text{and} \quad \{\theta_i\theta_j, \theta_j\theta_i\} = v^{(i,j)}\xi_j\xi_i,$$

where $\xi_i = \{\theta_i, \theta_i\}$ as before. In particular, we have that

$$0 \leq \{\theta_i\theta_j - v^{(i,j)}\theta_j\theta_i, \theta_i\theta_j - v^{(i,j)}\theta_j\theta_i\} = \xi_i\xi_j(1 - q^{(i,j)}),$$

where we have used that $v^2 = q$ together with the positive definiteness of the bilinear form. Since $q > 1$, we see that $(i, j) \leq 0$. Moreover, if $(i, j) = 0$, then $\theta_i\theta_j = \theta_j\theta_i$. This proves the first part.

Next, for all $\alpha \in \mathbb{Z}Q_0$ we have $(\alpha, \alpha) = 2\langle\alpha, \alpha\rangle \in 2\mathbb{Z}$, hence $(i, i) \in 2\mathbb{Z}$ for all $i \in I$. If $i \notin Q_0$, then for all $j \in Q_0$ we have $(\underline{\dim}\,\theta_i, e_j) = (i, j) \leq 0$. Since $\underline{\dim}\,\theta_i \in \mathbb{Z}Q_0$ is a linear combination of the e_j with non-negative coefficients, it follows that $(\underline{\dim}\,\theta_i, \underline{\dim}\,\theta_i) = (i, i) \leq 0$, proving the second part.

Finally, if $i \in Q_0$, then $\frac{1}{s_i}(e_i, e_j) \in \mathbb{Z}$ for all $j \in Q_0$ by the results in Section 5. It follows that $\frac{1}{s_i}(i, j) = \frac{1}{s_i}(e_i, \underline{\dim}\,\theta_j) \in \mathbb{Z}$, since it is a linear combination of integers. □

COROLLARY 2.23. *The Ringel-Hall algebra* H *is generated by the θ_i. This induces a $\mathbb{Z}I$-grading, where* $\deg\theta_i = i$*, which is a refinement of the original $\mathbb{Z}Q_0$-grading.*

We call I the set of simple roots. This splits into two sets, called the real and imaginary simple roots, where i is real if $(i,i) > 0$ and imaginary otherwise. Note that i is real if and only if $i \in Q_0$ and there are no vertex loops at i.

CHAPTER 3

Relation to Quantum Groups and Kac's Theorem

1. Borcherds Lie Algebras

In this section we present some of the main results from the theory of symmetrisable Borcherds Lie algebras. It is not intended to be a complete guide, and (at the moment) we only briefly mention the category $\mathcal{O}$. The main references are the books by Kac and Carter.

1.1. Symmetrisable Borcherds Matrices. A symmetrisable Borcherds matrix is given by integer matrices $C = D^{-1}B$ indexed by some countable set I such that $D = \mathrm{diag}(s_i)$ is diagonal with $s_i \geq 1$, B is symmetric with $b_{ij} \leq 0$ for $i \neq j$, and $c_{ii} \in 2\mathbb{Z}$ with $c_{ii} \leq 2$. We write

$$I^{\mathrm{re}} := \{i : c_{ii} = 2\} \quad \text{and} \quad I^{\mathrm{im}} := \{i : c_{ii} \leq 0\}. \tag{1.1}$$

We saw in Section 5 how to attach a valued graph to a finite symmetrisable Borcherds matrix, and that each such arises via the Euler form of an hereditary algebra over a finite field and valued graphs. A symmetrisable generalised Cartan matrix is a finite symmetrisable Borcherds matrix such that $I = I^{\mathrm{re}}$.

A realisation of C is given by a $\mathbb{Q}$-vector space $\mathfrak{h}$ of countable dimension equipped with a non-degenerate symmetric bilinear form $\{-,-\}$ together with a linearly independent set $\Pi^\vee = \{H_i : i \in I\}$ such that $\{H_i, H_j\} = b_{ij}/s_is_j$. Such a realisation always exists. Moreover, by enlarging $\mathfrak{h}$, we may further assume that there exists an element H_ρ such that $\{H_\rho, H_i\} = c_{ii}/2$ for all $i \in I$.

We use $\{-,-\}$ to identify $\mathfrak{h}$ with a subspace $\mathfrak{h}_0^*$ of its dual $\mathfrak{h}^*$, which then has a non-degenerate symmetric bilinear form $(-,-)$ by transport of structure. We set $e_i := s_i\{H_i, -\}$ and $\rho := \{H_\rho, -\}$, and call $\Pi = \{e_i : i \in I\}$ the set of simple roots. In summary,

$$\begin{aligned} \{H_i, H_j\} &= b_{ij}/s_is_j \\ e_j(H_i) &= b_{ij}/s_i = c_{ij} \\ (e_i, e_j) &= b_{ij} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \rho(H_i) &= \{H_\rho, H_i\} = c_{ii}/2 \\ e_i(H_\rho) &= (e_i, \rho) = b_{ii}/2. \end{aligned} \tag{1.2}$$

We identify the lattice inside $\mathfrak{h}^*$ generated by the e_i with $\mathbb{Z}I$ and call it the root lattice.

We recall that there exists a pair of bases $\{H_b : b \in B\}$ and $\{H^b : b \in B\}$ of $\mathfrak{h}$ such that $\{H_b, H^c\} = \delta_{b,c}$. For any such pair of bases, we have

$$\sum_{b \in B} \lambda(H_b)\lambda(H^b) = (\lambda, \lambda) \quad \text{for all } \lambda \in \mathfrak{h}_0^*. \tag{1.3}$$

For $\alpha = \sum_i \alpha_i e_i \in \mathbb{Z}I$ we define its support as

$$\mathrm{supp}(\alpha) := \{i \in I : \alpha_i \neq 0\} \subset I. \tag{1.4}$$

Also, given a subset $J \subset I$, we call J disconnected if there exists a non-trivial decomposition $J = J_1 \cup J_2$ into disjoint subsets such that $(e_{j_1}, e_{j_2}) = 0$ for all $j_r \in J_r$. We remark that, using the correspondence between finite symmetrisable Borcherds matrices and valued graphs, a subset $J \subset I$ is connected in the above sense if and only if it is the set of vertices of a connected subgraph.

For each $i \in I^{\mathrm{re}}$ we define the simple reflection r_i on $\mathfrak{h}^*$ via

$$r_i(\alpha) := \alpha - \frac{1}{s_i}(\alpha, e_i)e_i. \tag{1.5}$$

The Weyl group W is the group of automorphisms generated by the simple reflections. It is clear that the Weyl group preserves both $\mathfrak{h}_0^*$ and its bilinear form, and that it restricts to an automorphism of the root lattice $\mathbb{Z}I$.

1.2. Borcherds Lie Algebras. Let $\mathfrak{h}$ be a realisation of C.

Definition 3.1. We define $\tilde{g}$ to be the Lie algebra generated by $\mathfrak{h}$ and elements E_i and F_i for $i \in I$ such that $\mathfrak{h}$ is an abelian Lie subalgebra and

$$[E_i, F_j] = \delta_{ij} H_i, \quad [H, E_j] = e_j(H) E_j, \quad [H, F_j] = -e_j(H) F_j \tag{1.6}$$

for all $H, H' \in \mathfrak{h}$, $i, j \in I$.

Proposition 3.2. *There is a triangular decomposition* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{g}}^+$, *where* $\tilde{\mathfrak{g}}^+$ *(respectively* $\tilde{\mathfrak{g}}^-$*) is the free Lie algebra generated by the* E_i *(respectively the* F_i*). Moreover,* $\tilde{\mathfrak{g}}$ *is a* $\mathbb{Z}I$*-graded Lie algebra, where*

$$\deg E_i = e_i = -\deg F_i \quad \text{and} \quad \deg H = 0.$$

Since the form is non-degenerate, we see that

$$\tilde{\mathfrak{g}}_\alpha := \{x \in \tilde{\mathfrak{g}} : [H, x] = \alpha(H) x \quad \text{for all } H \in \mathfrak{h}\}. \tag{1.7}$$

We remark that all ideals $\mathfrak{r}$ of $\tilde{\mathfrak{g}}$ are graded. We write $\mathfrak{r}^\pm := \mathfrak{r} \cap \tilde{\mathfrak{g}}^\pm$ and $\mathfrak{r}^0 := \mathfrak{r} \cap \mathfrak{h}$.

We now endow $\tilde{\mathfrak{g}}$ with a symmetric bilinear form $\{-,-\}$, which is invariant in the sense that

$$\{x, [y, z]\} = \{[x, y], z\}, \tag{1.8}$$

and which extends the bilinear form on $\mathfrak{h}$.

Theorem 3.3. *There is a unique invariant symmetric bilinear form on* $\tilde{\mathfrak{g}}$ *extending the symmetric bilinear form on* $\tilde{\mathfrak{h}}$. *In particular, this satisfies*

$$\{\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta\} = 0 \quad \textit{unless } \alpha + \beta = 0 \quad \textit{and} \quad \{E_i, F_i\} = 1/s_i.$$

Proof. Let $\tilde{\mathfrak{g}}(r)$ be the subspace of $\tilde{\mathfrak{g}}$ spanned by all Lie monomials of length at most r. These clearly induce a filtration of $\tilde{\mathfrak{g}}$.

We start by using the invariance to construct some necessary relations. For $x \in \tilde{\mathfrak{g}}_\alpha$, $y \in \tilde{\mathfrak{g}}_\beta$ and $H \in \mathfrak{h}$ we have

$$\beta(H)\{x, y\} = \{x, [H, y]\} = \{[x, H], y\} = -\alpha(H)\{x, y\}.$$

It follows that $\{\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta\} = 0$ unless $\alpha + \beta = 0$.

We introduce the notation

$$[E_{i_1} \cdots E_{i_r}] := [E_{i_1}, [\cdots, [E_{i_{r-1}}, E_{i_r}]]]. \tag{1.9}$$

Then, for two Lie monomials $[E_{i_1} \cdots E_{i_r}]$ and $[F_{j_1} \cdots F_{j_r}]$, we have

$$\begin{aligned}
\{[E_{i_1} \cdots E_{i_r}], [F_{j_1} \cdots F_{j_r}]\} &= -\{[E_{i_2} \cdots E_{i_r}], [E_{i_1} F_{j_1} \cdots F_{j_r}]\} \\
&= -\sum_{s=1}^{r} \delta_{i_1 j_s} \{[E_{i_2} \cdots E_{i_r}], [F_{j_1} \cdots H_{j_s} \cdots F_{j_r}]\} \\
&= \sum_{s=1}^{r} \delta_{i_1 j_s} (c_{j_s j_{s+1}} + \cdots + c_{j_s j_r}) \{[E_{i_2} \cdots E_{i_r}], [F_{j_1} \cdots \hat{F}_{j_s} \cdots F_{j_r}]\}.
\end{aligned}$$

Thus the definition of the bracket product on Lie monomials of length r is completely determined by the definition of those of length $r-1$. Finally, if $e_i(H) \neq 0$, then

$$e_i(H)\{E_i, F_i\} = \{E_i, [F_i, H]\} = \{[E_i, F_i], H\} = \{H_i, H\} = e_i(H)/s_i.$$

Thus $\{E_i, F_i\} = 1/s_i$ and so there is at most one extension of the bilinear form to the whole of $\tilde{\mathfrak{g}}$.

We can use the above relations to define a form on pairs of Lie monomials. That this defines a bilinear form on $\tilde{\mathfrak{g}}$ follows from the invariance on smaller Lie monomials together with the Jacobi identity. We deduce that the form is invariant on $\tilde{\mathfrak{g}}$. The details can be found in either [Kac] or [Carter]. □

We denote the radical of this form by

$$\mathrm{rad} := \{x \in \tilde{\mathfrak{g}} : \{x, \tilde{\mathfrak{g}}\} = 0\}.$$

Then rad is a graded Lie ideal intersecting $\mathfrak{h}$ trivially.

Lemma 3.4. *The ideal* rad *is the unique maximal (graded) ideal intersecting* $\mathfrak{h}$ *trivially.*

Proof. Let $\mathfrak{r}$ be any ideal intersecting $\mathfrak{h}$ trivially and suppose that $\mathfrak{r}_\beta \subset \mathrm{rad}$ for all $0 \leq \beta < \alpha$. Consider $x \in \mathfrak{r}_\alpha$. Then for all $i \in I$ such that $\alpha_i > 0$, $[x, F_i] \in \mathfrak{r}_{\alpha - e_i} \subset \mathrm{rad}$. Thus $\{x, [F_i, y]\} = \{[x, F_i], y\} = 0$, and since $\tilde{\mathfrak{g}}_{-\alpha}$ is spanned by the elements $[F_i, y]$ with $y \in \tilde{\mathfrak{g}}_{e_i - \alpha}$, we see that $x \in \mathrm{rad}$. □

Definition 3.5. The Borcherds Lie algebra associated to the realisation $\mathfrak{h}$ of C is the quotient $\mathfrak{g} := \tilde{\mathfrak{g}}/\mathrm{rad}$. The bilinear form clearly descends to an invariant symmetric bilinear form on $\mathfrak{g}$, and this form induces a non-degenerate pairing

$$\{-, -\} \colon \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{Q}$$

for all $\alpha \in \mathbb{Z}Q_0$.

It follows immediately that $\mathfrak{g}$ is again $\mathbb{Z}I$-graded with $\mathfrak{g}_0 = \mathfrak{h}$. Moreover, since rad intersects $\mathfrak{h}$ trivially, we see that rad^+ is again an ideal and $\mathfrak{g}^+ = \tilde{\mathfrak{g}}^+/\mathrm{rad}^+$.

Proposition 3.6. *For* $i \neq j \in I$ *we have*

$$\{(\mathrm{ad}\, E_i)^n(E_j), (\mathrm{ad}\, F_i)^n(F_j)\} = (-1)^n \frac{n!}{s_j} \prod_{r=0}^{n-1} (c_{ij} + r c_{ii}/2).$$

In particular, the following Serre elements all lie in the radical:

$$\begin{gathered} (\mathrm{ad}\, E_i)^{1-c_{ij}}(E_j) \quad \textit{and} \quad (\mathrm{ad}\, F_i)^{1-c_{ij}}(F_j) \quad \textit{if } i \in I^{\mathrm{re}} \textit{ and } j \neq i \\ [E_i, E_j] \quad \textit{and} \quad [F_i, F_j] \quad \textit{if } c_{ij} = 0. \end{gathered} \tag{1.10}$$

Proof. We begin by observing that

$$\begin{aligned} (\mathrm{ad}\, E_i)(\mathrm{ad}\, F_i)^n(F_j) &= \sum_{r+s=n-1} (\mathrm{ad}\, F_i)^r (\mathrm{ad}\, H_i)(\mathrm{ad}\, F_i)^s (F_j) \\ &= -\sum_{r+s=n-1} (c_{ij} + s c_{ii})(\mathrm{ad}\, F_i)^{n-1}(F_j) \\ &= -n(c_{ij} + (n-1)c_{ii}/2)(\mathrm{ad}\, F_i)^{n-1}(F_j). \end{aligned}$$

The first statement now follows by induction on n, using the invariance of the bilinear form. In particular, the Serre relations correspond to elements in the radical of the bilinear form, since $\tilde{\mathfrak{g}}_{ne_i + e_j}$ is one dimensional. The last statment is now clear. □

Definition 3.7. Let $J \subseteq \mathrm{rad}$ be the ideal of $\tilde{\mathfrak{g}}$ generated by the Serre elements defined above in Equation (1.10). We define the Lie algebra $\overline{\mathfrak{g}} := \tilde{\mathfrak{g}}/J$. Then $\overline{\mathfrak{g}}$ is again $\mathbb{Z}I$-graded with $\overline{\mathfrak{g}}_0 = \mathfrak{h}$ and the invariant bilinear form on $\tilde{\mathfrak{g}}$ descends to $\overline{\mathfrak{g}}$.

We note some immediate consequences of the definitions. There is an epimorphism of graded Lie algebras $\overline{\mathfrak{g}} \to \mathfrak{g}$ which respects the bilinear forms. Moreover, $\mathrm{ad}\, E_i$ and $\mathrm{ad}\, F_i$ for $i \in I^{\mathrm{re}}$ act locally nilpotently on both $\overline{\mathfrak{g}}$ and $\mathfrak{g}$.

The main result that we require is that $\overline{\mathfrak{g}} = \mathfrak{g}$, or equivalently that $J = \mathrm{rad}$. This we show in Theorem 3.11.

The roots of $\mathfrak{g}$ are defined to be

$$\Phi := \{\alpha \in \mathbb{Z}I : \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}. \tag{1.11}$$

This has a completely combinatorial description, which we now give.

The fundamental region is defined as

$$K := \{\alpha > 0 : (\alpha, e_i) \leq 0 \text{ for all } i \in I, \operatorname{supp}(\alpha) \text{ connected}\} \setminus \{ne_i : i \in I^{\mathrm{im}}, n \geq 2\}. \tag{1.12}$$

Note that if $\alpha > 0$ and $i \in I^{\mathrm{im}}$, then we necessarily have that $(\alpha, e_i) \leq 0$. In particular, $e_i \in K$ for all $i \in I^{\mathrm{im}}$. Conversely, we know that $\mathfrak{g}_{ne_i} = 0$ for all $i \in I$ and all $n \geq 2$, hence $ne_i \notin \Phi$ for all $n \geq 2$.

Theorem 3.8. *The set of roots decomposes into real and imaginary roots* $\Phi = \Phi^{\mathrm{re}} \cup \Phi^{\mathrm{im}}$, *where*

$$\Phi^{\mathrm{re}} := \bigcup_{i \in I^{\mathrm{re}}} We_i \quad \text{and} \quad \Phi^{\mathrm{im}} = \pm \bigcup_{\alpha \in K} W\alpha.$$

Proof. We know that $\mathrm{ad}\, E_i$ and $\mathrm{ad}\, F_i$ for $i \in I^{\mathrm{re}}$ act locally nilpotently on $\mathfrak{g}$. It follows that the expression

$$t_i := \exp(\mathrm{ad}\, E_i) \exp(-\mathrm{ad}\, F_i) \exp(\mathrm{ad}\, E_i) \tag{1.13}$$

is a well-defined automorphism of $\mathfrak{g}$ for all $i \in I^{\mathrm{re}}$. It is easy to check that

$$t_i(H) = H - \frac{1}{s_i} e_i(H) H_i \quad \text{for all } H \in \mathfrak{h}. \tag{1.14}$$

This action is dual to the reflection r_i on $\mathfrak{h}^*$. In particular, we have an induced isomorphism $t_i \colon \mathfrak{g}_\alpha \xrightarrow{\sim} \mathfrak{g}_{r_i(\alpha)}$, and so the set of roots Φ is W-invariant.

Clearly all simple roots are contained in Φ, and every positive root is of the form $w(\alpha)$ for some $\alpha \in \Pi \cup K$. It remains to show that all roots in the fundamental region K are contained in Φ. We refer the reader to [Kac] or [Carter].

In fact, with some extra work, one can show that the automorphisms t_i describe an action of the braid group on $\mathfrak{g}$ (reference?). □

We define the root multiplicites as

$$\mathrm{mult}(\alpha, \mathfrak{g}) := \dim \mathfrak{g}_\alpha. \tag{1.15}$$

It follows that

$$\mathrm{mult}(\alpha, \mathfrak{g}) = \mathrm{mult}(r_i(\alpha), \mathfrak{g}) \quad \text{for all } i \in I^{\mathrm{re}} \text{ and } \alpha \in \mathbb{Z}I. \tag{1.16}$$

1.3. Category $\mathcal{O}$. We now describe the category $\mathcal{O}$, a certain subcategory of (left) $\mathfrak{g}$-modules satisfying certain nice properties.

Let M be a $\mathfrak{g}$-module. For $\lambda \in \mathfrak{h}^*$ the space

$$M_\lambda := \{m \in M : H \cdot m = \lambda(H)m \text{ for all } H \in \mathfrak{h}\}$$

is called the λ-weight space of M.

A module is called $\mathfrak{h}$-diagonalisable if it is the direct sum of its weight spaces. If M is $\mathfrak{h}$-diagonalisable, then we define its support to be

$$\mathrm{supp}(M) := \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}.$$

Some important $\mathfrak{h}$-diagonalisable modules are the lowest weight Verma modules. Given $\lambda \in \mathfrak{h}_0^*$ let $\mathbb{Q}_\lambda$ be the one-dimensional $U(\mathfrak{h} \oplus \mathfrak{g}^-)$-module with basis vector 1_λ such that

$$H \cdot 1_\lambda := \lambda(H)1_\lambda \quad \text{and} \quad F_i \cdot 1_\lambda := 0. \tag{1.17}$$

The Verma module $M(\lambda)$ is defined to be the induced module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{g}^-)} \mathbb{Q}_\lambda. \tag{1.18}$$

As a vector space, we see that

$$M(\lambda)_\mu \cong U(\mathfrak{g}^+)_{\mu-\lambda}, \tag{1.19}$$

and so

$$\mathrm{supp}(M(\lambda)) = \lambda + \mathbb{N}_0 I. \tag{1.20}$$

We are now in a position to define the category $\mathcal{O}$. Its objects are the $\mathfrak{h}$-diagonalisable modules M with finite dimensional weight spaces and whose support is contained in the supports of finitely many lowest weight Verma modules. That is, there exists $\lambda_1, \ldots, \lambda_n \in \mathfrak{h}_0^*$ such that

$$\mathrm{supp}(M) \subseteq \mathrm{supp}\big(\bigoplus_i M(\lambda_i)\big) = \bigcup_i \big(\lambda_i + \mathbb{N}_0 I\big).$$

It follows immediately that each F_i acts locally nilpotently on each $M \in \mathcal{O}$.

Given $M \in \mathcal{O}$, a non-zero vector $v \in M_\lambda$ in some weight space is called a heighest weight vector if $v \neq 0$ but $\mathfrak{g}^- \cdot v = 0$. It follows that there is a morphism $M(\lambda) \to M$ sending 1_λ to v. More generally, v is called a primitive vector if it is a heighest weight vector in some factor module of M.

Lemma 3.9. *Each module $M \in \mathcal{O}$ is generated (even as a $U(\mathfrak{g}^+)$-module) by its primitive vectors.*

Given a module $M \in \mathcal{O}$, we define its formal character as

$$\mathrm{ch}(M) := \sum_\lambda (\dim M_\lambda) e^\lambda. \tag{1.21}$$

Defining $e^\lambda e^\mu := e^{\lambda+\mu}$, we see that the character can be veiwed as an element in a certain completion of the group ring of $\mathfrak{h}^*$ (as an additive group). It follows easily from the Poincaré-Birkhoff-Witt Theorem that

$$\mathrm{ch}(M(\lambda)) = e^\lambda \mathrm{ch}(U(\mathfrak{g})^+) = e^\lambda \prod_{\alpha\in\Phi^+} (1-e^\alpha)^{-\mathrm{mult}(\alpha,\mathfrak{g})}. \tag{1.22}$$

The Casimir operator Ω is of central importance in studying the category $\mathcal{O}$. It is defined as follows. Take a homogeneous basis $\{E_a : a \in A\}$ of $\mathfrak{g}^+$ and take the dual basis $\{F_a : a \in A\}$ with respect to the non-degenerate form $\{-,-\}$. Next take a pair of bases $\{H_b : b \in B\}$ and $\{H^b : b \in B\}$ of $\mathfrak{h}$ which are dual with respect to $\{-,-\}$. Recall also the element $H_\rho \in \mathfrak{h}$ such that $\{H_\rho, H_i\} = c_{ii}/2$ for all $i \in I$. Then the Casimir operator is given by

$$\Omega := \sum_b H^b H_b - 2H_\rho + 2\sum_a E_a F_a. \tag{1.23}$$

Given $M \in \mathcal{O}$ and $v \in M_\lambda$, we have $F_a \cdot v = 0$ for almost all $a \in A$, and $\sum_b H^b H_b \cdot v = (\lambda, \lambda)v$. Thus Ω is well-defined on each $M \in \mathcal{O}$.

Lemma 3.10. *The operator Ω is independent of the choices of bases and commutes with the action of $U(\mathfrak{g})$.*

Using these facts we see that Ω acts on the Verma module $M(\lambda)$ for $\lambda \in \mathfrak{h}_0^*$ as scalar multiplication by $(\lambda, \lambda - 2\rho) = (\lambda - \rho, \lambda - \rho) - (\rho, \rho)$. For,

$$\begin{aligned}\Omega \cdot 1_\lambda &= \sum_b H^b \cdot H_b \cdot 1_\lambda - 2H_\rho \cdot 1_\lambda \\ &= \big(\sum_b \lambda(H^b)\lambda(H_b) - 2\lambda(H_\rho)\big)1_\lambda \\ &= \big((\lambda, \lambda) - 2(\lambda, \rho)\big)1_\lambda.\end{aligned}$$

We note that Verma modules can also be defined for both $\tilde{\mathfrak{g}}$ and $\overline{\mathfrak{g}}$. In fact, as before, write $\mathbb{Q}_\lambda$ for the one dimensional $U(\mathfrak{h} \oplus \tilde{\mathfrak{g}}^-)$-module with basis vector 1_λ such that $H \cdot 1_\lambda = \lambda(H)1_\lambda$ and $F_i \cdot 1_\lambda = 0$, and define

$$\widetilde{M}(\lambda) := U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{h} \oplus \tilde{\mathfrak{g}}^-)} \mathbb{Q}_\lambda.$$

Then

$$\begin{aligned}\overline{M}(\lambda) &= U(\overline{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}})} \widetilde{M}(\lambda) \\ M(\lambda) &= U(\mathfrak{g}) \otimes_{U(\tilde{\mathfrak{g}})} \widetilde{M}(\lambda)\end{aligned}$$

are the Verma modules for $\overline{\mathfrak{g}}$ and $\mathfrak{g}$.

On the other hand, the definition of the Casimir operator required the non-degeneracy of the form $\{-,-\}$, and for that reason cannot *a priori* be lifted to $\overline{\mathfrak{g}}$ or $\tilde{\mathfrak{g}}$.

1.4. A presentation for Borcherds Lie algebras. We can now sketch the proof of the following theorem, which shows that $\overline{\mathfrak{g}} = \mathfrak{g}$.

Theorem 3.11. *The radical is generated as an ideal of $\tilde{\mathfrak{g}}$ by the Serre elements*

$$\begin{gathered}(\operatorname{ad} E_i)^{1-c_{ij}}(E_j) \quad \text{and} \quad (\operatorname{ad} F_i)^{1-c_{ij}}(F_j) \quad \text{for } i \in I^{re} \text{ and } j \neq i, \\ [E_i, E_j] \quad \text{and} \quad [F_i, F_j] \quad \text{whenever } c_{ij} = 0.\end{gathered}$$

Proof. Recall that J is the ideal of $\tilde{\mathfrak{g}}$ generated by the above elements. We have already shown that $J \subset \mathrm{rad}$ and since everything respects the grading we can write $\mathrm{rad}/J = \bigoplus_{\alpha \in \mathbb{Z}I} \mathrm{rad}_\alpha / J_\alpha$.

We have already noted that rad^+ is an ideal of $\tilde{\mathfrak{g}}$, and using the isomorphism $\widetilde{M}(0) \cong U(\tilde{\mathfrak{g}}^+)$ as graded vector spaces, we obtain an embedding $\mathrm{rad}^+ \to \widetilde{M}(0)$. This is naturally an embedding of $\tilde{\mathfrak{g}}$-modules.

Next, since $U(\tilde{\mathfrak{g}}^+)$ is a free algebra, the natural $\tilde{\mathfrak{g}}$-homomorphism

$$\bigoplus_{i \in I} \widetilde{M}(e_i) \to \widetilde{M}(0), \qquad (x_i \cdot 1_{e_i})_i \mapsto \sum_i x_i E_i \cdot 1_0$$

is an embedding. Since this is of codimension one, $\bigoplus_i \widetilde{M}(e_i)$ must be the unique maximal submodule of $\widetilde{M}(0)$. We thus obtain the map

$$\mathrm{rad}^+ \to \bigoplus_{i\in I} \widetilde{M}(e_i) \subset \widetilde{M}(0)$$

and hence a map

$$\mathrm{rad}^+ \to \bigoplus_{i\in I} \widetilde{M}(e_i) \to \bigoplus_{i\in I} M(e_i).$$

The kernel of the map $\widetilde{M}(e_i) \to M(e_i)$ is precisely the kernel of the map $U(\tilde{\mathfrak{g}}^+) \to U(\mathfrak{g}^+)$, which equals $\mathrm{rad}^+ U(\tilde{\mathfrak{g}}^+)$. Thus the kernel of the map $\mathrm{rad}^+ \to \bigoplus_i M(e_i)$ equals

$$\mathrm{rad}^+ \cap \mathrm{rad}^+ U(\tilde{\mathfrak{g}}^+) = [\mathrm{rad}^+, \mathrm{rad}^+] =: \mathfrak{c}.$$

The commutator $\mathfrak{c}$ is a small submodule of rad^+, since if $\mathfrak{r}$ is a submodule such that $\mathfrak{r}+\mathfrak{c} = \mathrm{rad}^+$, then $\mathfrak{c}/(\mathfrak{c}\cap\mathfrak{r}) = \mathrm{rad}^+/\mathfrak{r}$ and so $\mathrm{rad}^+/\mathfrak{r}$ equals its own commutator. This, however, is impossible by considering minimal weights.

In particular, we deduce that rad^+ is generated by those weight spaces corresponding to the weights of the primitive vectors of $\mathrm{rad}^+/\mathfrak{c}$. Since the Casimir operator acts as zero on each $M(e_i)$, we see that each primitive vector of $\mathrm{rad}^+/\mathfrak{c}$ has weight λ satisfying $(\lambda,\lambda) = 2(\rho,\lambda)$. It follows that the quotient rad^+/J is generated by its weight spaces of weight λ satisfying $(\lambda,\lambda) = 2(\rho,\lambda)$.

Now suppose that $(\mathrm{rad}^+/J)_\alpha \neq 0$ with α minimal. We know that the Serre relations hold in $\tilde{\mathfrak{g}}/J$, so $\mathrm{ad}\,(E_i)$ and $\mathrm{ad}\,F_i$ for $i \in I^{\mathrm{re}}$ act locally nilpotently. We can thus define the automorphism t_i for $i \in I^{\mathrm{re}}$ as in the proof of Theorem 3.8. In particular, since

$$\dim(\mathrm{rad}/J)_\alpha = \dim \mathfrak{g}_\alpha - \dim(\tilde{\mathfrak{g}}/J)_\alpha,$$

we see that the Weyl group acts on the set $\{\alpha : (\mathrm{rad}/J)_\alpha \neq 0\}$. By the minimality of α, we see that $r_i(\alpha) \geq \alpha$ for all $i \in I^{\mathrm{re}}$, and hence $(\alpha, e_i) \leq 0$ for all $i \in I$. In particular, $(\alpha,\alpha) \leq 0$.

Putting this together, if $\alpha = \sum_i \alpha_i e_i \geq 0$ is minimal such that $\mathrm{rad}^+_\alpha/J^+_\alpha \neq 0$, then $(\alpha, e_i) \leq 0$ for all i, and $(\alpha,\alpha) = 2(\rho,\alpha)$, or equivalently, $\sum_i(\alpha - e_i, e_i)\alpha_i = 0$.

Now, if $i \in I^{\mathrm{re}}$, then $(e_i, e_i) = b_{ii} > 0$, so that $(\alpha - e_i, e_i) < 0$. On the other hand, if $i \in \mathrm{supp}(\alpha) \cap I^{\mathrm{im}}$, then

$$(\alpha - e_i, e_i) = (\alpha_i - 1)b_{ii} + \sum_{j\neq i} \alpha_j (e_j, e_i) \leq 0.$$

We deduce that $\alpha_i = 0$ for all $i \in I^{\mathrm{re}}$, that $(e_i, e_j) = 0$ if $i, j \in \mathrm{supp}(\alpha)$, and that $(e_i, e_i) = 0$ if $\alpha_i \geq 2$. For such a weight α we note that $\tilde{\mathfrak{g}}_\alpha = 0$, since $[E_i, E_j] = 0$ for all $i, j \in \mathrm{supp}(\alpha)$. Hence $J^+_\alpha = \mathrm{rad}^+_\alpha = 0$, a contradiction. □

It now follows that the universal enveloping algebra $U(\mathfrak{g})$ is the quotient of $U(\tilde{\mathfrak{g}})$ by the ideal generated by the Serre elements

$$\sum_{r+s=1-c_{ij}} (-1)^r E_i^{(r)} E_j E_i^{(s)} = 0 \quad \text{and} \quad \sum_{r+s=1-c_{ij}} (-1)^r F_i^{(r)} F_j F_i^{(s)} \quad \text{for } i \neq j,$$

$$E_iE_j = E_jE_i \quad \text{and} \quad F_iF_j = F_jF_i \quad \text{whenever } c_{ij} = 0, \tag{1.24}$$

where we have used the divided powers $E_i^{(r)} = \frac{1}{r!}E_i^r$. Moreover, $\mathfrak{g}^+ = \tilde{\mathfrak{g}}/\mathrm{rad}^+$ has a presentation via the generators E_i for $i \in Q_0$ and the relations

$$\begin{aligned}(\operatorname{ad} E_i)^{1-c_{ij}}(E_j) &= 0 \quad \text{for } i \neq j,\\ [E_i, E_j] &= 0 \quad \text{whenever } c_{ij} = 0.\end{aligned}$$

Thus $U(\mathfrak{g}^+)$ has a presentation via the generators E_i for $i \in Q_0$ and the relations

$$\begin{aligned}\sum_{r+s=1-c_{ij}} (-1)^r E_i^{(r)} E_j E_i^{(s)} &= 0 \quad \text{for } i \neq j,\\ E_i E_j &= E_j E_i \quad \text{whenever } c_{ij} = 0.\end{aligned}$$