Commutative algebra and algebraic geometry
Exercises 2

1. The Jacobson radical of a ring R is defined to be the intersection of all maximal ideals,

$$J(R) := \bigcap_{\text{max ideals}} I.$$

In particular, $J(R)$ is a radical ideal.

Show that $x \in J(R)$ if and only if $1 + rx$ is invertible for all $r \in R$.

2. Let R be a ring, $a \in R$, and set $\Sigma := \{1, a, a^2, a^3, \ldots\}$. Note that Σ is a multiplicatively closed subset. Prove that the localisation $R_a := R_\Sigma$ is isomorphic to $R[X]/(aX - 1)$, a quotient of the polynomial ring.

3. Let $\Sigma \subset R$ be multiplicatively closed. Show that the following defines an equivalence relation on $R \times \Sigma$

$$(r, a) \sim (s, b) \quad \text{provided there exists } c \in \Sigma \text{ with } (sa - rb)c = 0.$$

Show further that the sum and product on the set of equivalence classes R_Σ is well-defined, thus proving Lemma 3.2.

4. Make sure you know the statements and proofs of the three isomorphism theorems for modules.

Prove that if $U \leq M$ are R-modules, and $I \lhd R$ an ideal, then $I(M/U) = (IM + U)/U$.

5. Let $N \leq M$ be R-modules, and suppose that M/N is finitely generated. Show that, if $I \subset J(R)$ and $N + IM = M$, then $N = M$.

Now let R be a local ring, with unique maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = K$. One usually writes this as $(R, \mathfrak{m}, K)$. Let M be a finitely generated R-module.

Show that $M/\mathfrak{m}M$ is naturally a K-vector space. Show further that $x_1, \ldots, x_n$ generate M if and only if their images $\bar{x}_1, \ldots, \bar{x}_n$ span $M/\mathfrak{m}M$ as a K-vector space.

Can you find a counter-example when M is not finitely generated? Hint. Consider the local ring $\mathbb{Z}_{(p)}$.

Extra questions

6. The power series ring $K[\![X]\!]$ is the set of formal sums $\sum_{n\geq 0} a_n X^n$ with $a_n \in K$. (We say that this is a formal sum to emphasise that the coefficients a_n can be arbitrary.) The addition is done pointwise, and the multiplication is induced by $X^m X^n = X^{m+n}$.

   (a) Show that a power series $f = \sum_n a_n X^n$ is a unit if and only if $a_0 \neq 0$. Hence show every proper ideal of $K[\![X]\!]$ is of the form (X^r). In particular, $K[\![X]\!]$ is a principal ideal domain, has a unique maximal ideal (X), and precisely two prime ideals, (X) and (0). As such, it is a **discrete valuation ring**.

   (b) For $f = \sum_n f_n X^n$ in $K[\![X]\!]$ non-zero define $N(f) := 2^{-r}$ where r is minimal such that $f_r \neq 0$, and set $N(0) := 0$. Show that $d(f, g) := N(f - g)$ defines a **metric** on $K[\![X]\!]$. In other words, $d(f, g) \geq 0$ with equality if and only if $f = g$, and the triangle inequality holds: $d(f, g) \leq d(f, h) + d(g, h)$ for all h.

Show further that if we endow $K[\![X]\!] \times K[\![X]\!]$ with the product topology, then addition and multiplication are continuous as maps $K[\![X]\!] \times K[\![X]\!] \to K[\![X]\!]$. Hence $K[\![X]\!]$ is a **topological ring**.

Finally, show that $K[\![X]\!]$ is complete with respect to this topology; that is, every Cauchy sequence converges.

(c) Show that the natural map $K[X] \to K[\![X]\!]$ extends to an injective ring homomorphism $K[X]_{(X)} \to K[\![X]\!]$, where $K[X]_{(X)}$ is the localisation of $K[X]$ at the maximal ideal (X).

Show that $K[\![X]\!]$ is the completion of both of the subrings $K[X]$ and $K[X]_{(X)}$.

(d) Show that the map $K[X]_{(X)} \to K[\![X]\!]$ is not surjective, so in particular $K[X]_{(X)}$ is not complete.

Hint: Show that $p \in K[\![X]\!]$ is in the image of $K[X]_{(X)}$ if and only if there exists a monic polynomial g with pg a polynomial. Obtain a recurrence relation for the coefficients p_n for n sufficiently large. Now find some element p not satisfying any such condition.

7. Let M_i be a family of R-modules.

(a) Show that the i-th projection $\pi_i\colon \prod_j M_j \to M_i$ is an R-module homomorphism. Show further that these induce an isomorphism (of R-modules)

$$\mathrm{Hom}_R(X, \textstyle\prod_i M_i) \cong \prod_i \mathrm{Hom}_R(X, M_i), \quad f \mapsto (\pi_i f),$$

for all R-modules X. This proves that $\prod_i M_i$ is a categorical product.

(b) Show that the i-th embedding $\iota_i\colon M_i \to \coprod_j M_j$ is an R-module homomorphism. Show further that these induce an isomorphism (of R-modules)

$$\mathrm{Hom}_R(\textstyle\coprod_i M_i, X) \cong \prod_i \mathrm{Hom}_R(M_i, X), \quad f \mapsto (f\iota_i).$$

(c) Prove that for a set I and R-modules M and N, there is a (natural) isomorphism

$$\mathrm{Hom}_R(M^{(I)}, N) \cong \mathrm{Hom}_R(M, N^I).$$

In fact, the assignments $F\colon M \mapsto M^{(I)}$ and $G\colon N \mapsto N^I$ extend to an adjoint pair of functors (F, G).

8. Prove that the direct product of rings $R := \prod_i R_i$ is a categorical product; that is, there are natural ring homomorphisms $\pi_i\colon R \to R_i$ such that $\mathrm{Hom}(S, R) \to \prod_i \mathrm{Hom}(S, R_i)$, $f \mapsto (\pi_i f)$, is a bijection for all rings S.

Show that ring homomorphisms $\mathbb{Z} \times \mathbb{Z} \to R$ are in bijection with **idempotent** elements $x \in R$ (so satisfying $x^2 = x$), via $f \mapsto f(1, 0)$.

Show that in the category of rings the coproduct of $\mathbb{Z}$ with itself exists, and equals $\mathbb{Z}$.