Commutative algebra and algebraic geometry
Exercises 3

1. Prove Proposition 5.1.

2. Show that the following are equivalent for an R-module P.

   (a) P is projective.

   (b) Every epimorphism $g\colon M \twoheadrightarrow P$ has a section, so an R-linear $s\colon P \to M$ such that $gs = \mathrm{id}_P$.

   (c) There exists an R-module Q such that $P \oplus Q$ is free, so isomorphic to $R^{(I)}$ for some I.

   Dually, show that the following are equivalent for an R-module I.

   (a) I is injective.

   (b) Every monomorphism $f\colon I \rightarrowtail M$ has a retract, so an R-linear $r\colon M \to I$ such that $rf = \mathrm{id}_I$.

   Hint: for (b) implies (a), let $f\colon L \rightarrowtail M$ be a monomorphism. Given $h\colon L \to I$, consider the cokernel of $\binom{f}{h}\colon L \to M \oplus I$.

   > Remark: The third condition does not have an obvious dual; instead we have **Baer's Criterion**, which states that it is equivalent to take $M = R$ in (b). In other words, I is injective if and only if for all ideals $J \lhd R$, the map $\mathrm{Hom}_R(R, I) \to \mathrm{Hom}_R(J, I)$ is surjective.

3. Let (F, G) be an adjoint pair of functors. Show that the unit $\eta\colon \mathrm{id} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \mathrm{id}$ are natural transformations. This proves Lemma 6.1.

4. Let k be a commutative ring, and M, N and X three k-modules. A map $f\colon M \times N \to X$ is said to be **k-bilinear** provided, for each $m \in M$, the map $N \to X$, $n \mapsto f(m, n)$, is k-linear, and similarly for each $n \in N$, the map $M \to X$, $m \mapsto f(m, n)$ is k-linear. We write $\mathrm{Bil}_k(M \times N, X)$ for the set of all k-bilinear maps.

   (a) Prove that $\mathrm{Bil}_k(M \times N, X)$ is a naturally a k-module.

   (b) Prove that the map $\mathrm{Hom}_k(M, \mathrm{Hom}_k(N, X)) \to \mathrm{Bil}_k(M \times N, X)$, $\theta \mapsto \bar\theta$ where $\bar\theta(m, n) := \theta(m)(n)$, is an isomorphism of k-modules.

   (c) Prove that the map $\iota\colon M \times N \to M \otimes_k N$, $(m, n) \mapsto m \otimes n$, is k-bilinear.

   (d) Prove that the pair $(M \otimes_k N, \iota)$ satisfies the following universal property.
   For each k-module X and k-bilinear map $f\colon M \times N \to X$, there exists a unique k-linear map $\bar f\colon M \otimes_k N \to X$ such that $f = \bar f \iota$.

5. Let $m, n \in \mathbb{Z}$, with greatest common divisor d. Show that the map

$$(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}, \quad (a, b) \mapsto ab,$$

   is (well-defined and) bilinear. Show that it induces an isomorphism

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}.$$

Use this to give another proof that the following sequence is exact

$$\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{m\otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to 0.$$

When is the left most map injective?

Extra questions

6. Use Baer's Criterion to prove that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

   Now let R be a ring, which we may regard as a $\mathbb{Z}$-R-bimodule. Thus for all abelian groups X, there is a natural R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(R, X)$, $rf \colon s \mapsto f(sr)$ for $f \colon R \to X$ and $r, s \in R$.

   Prove that $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. Hint: tensor products.

7. One source of adjoint pairs of functors is to take G to be a **forgetful functor**, where we forget some of the structure of the original category; a left adjoint F, if it exists, is then a **free functor**. Note that these functors generally go between different types of categories, rather than between module categories over different rings.

   (a) Show that there is a functor from R-modules to sets, sending a module M to its underlying set. Show further that this has a left adjoint, which sends a set I to the free module $R^{(I)}$.

   As a special case we have the K-vector space $K^{(I)}$, such that linear maps $K^{(I)} \to V$ are determined by where they send a basis, and hence are in bijection with maps of sets $I \to V$.

   (b) Show that there is a functor from the category of fields to the category of integral domains with injective ring homomorphisms, sending a field K to its underlying integral domain. Show further that this has a left adjoint, sending an integral domain to its quotient field.