Commutative algebra and algebraic geometry
Exercises 4

1. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Show that if L and N are finitely generated, then so too is M. (Note that we basically proved this when proving Hilbert's Basis Theorem.)

2. Let $N_1, N_2 \leq M$ be R-modules. Show that if M/N_i are Noetherian, then so too is $M/(N_1 \cap N_2)$.

   Hint: consider the epimorphism $M/(N_1 \cap N_2) \twoheadrightarrow M/N_1$, compute its kernel, and use the Second Isomorphism Theorem.

3. Show that the ring $R := K[T_1, T_2, T_3, \ldots]/(T_1^1, T_2^2, T_3^3, \ldots)$ is not Noetherian, but has a unique prime ideal.

4. Let K be a field, and consider the ring

$$R := K[X, T_0, T_1, \ldots]/(XT_1 - T_0, XT_2 - T_1, XT_3 - T_2, \ldots).$$

   (a) Show that the ideal (X) is maximal, and that the ideal $I := (T_0, T_1, T_2, \ldots)$ is prime. (Compute the quotient rings.)

   (b) Show that $I \subset \bigcap_n (X^n)$. Prove that $I = \bigcap_n (X^n)$ (Use the Krull Intersection Theorem on the quotient R/I.)

   (c) Consider the ring homomorphism $R \to K[X, X^{-1}, Y]$ sending $X \mapsto X$ and $T_i \mapsto X^{-i}Y$. Show that the image is the subring S consisting of all elements of the form $f(X) + g(X,Y)Y$ with $f(X) \in K[X]$ and $g(X,Y) \in K[X, X^{-1}, Y]$. Show that this is an isomorphism. (Construct a linear map $S \to R$ giving an isomorphism of vector spaces.)

5. (a) Let R be a ring, and I a proper ideal. Show that

$$\{a \in R : a(1-x) = 0 \text{ for some } x \in I\} \subset \bigcap_n I^n$$

   with equality when R is Noetherian.

   (b) Now consider the two properties for a ring R.

   (i) $\bigcap_n I^n = 0$ for all proper ideals I.

   (ii) Every zero divisor lies in $\mathrm{Jac}(R)$.

   Show that (i) implies (ii) always, and that (ii) implies (i) when R is Noetherian.

Extra questions

6. Let $\Sigma \subset R$ be multiplicatively closed. Show that $\mathrm{nil}(R_\Sigma) = \mathrm{nil}(R)_\Sigma$.

7. (a) Let M, N be flat R-modules. Show that $M \otimes_R N$ is also a flat R-module.

   (b) Let $R \to S$ be a ring homomorphism. Show that if M is a flat R-module, then $S \otimes_R M$ is a flat S-module.

8. Let $(R, \mathfrak{m}, K)$ be a local ring, and M, N two finitely generated R-modules. Show that $M \otimes_R N = 0$ implies $M = 0$ or $N = 0$.

   Hint: Tensor over K, using Lemma 8.2 and Exercise 2.5.