Commutative algebra and algebraic geometry
Exercises 5

1. Let R be a unique factorisation domain. Show that R is integrally closed. In particular, the polynomial algebra $K[X_1, \dots, X_n]$ over a field K is integrally closed, as is every principal ideal domain.

2. Set $R := K[X, Y]/(Y^2 - X^3)$.

   (a) Show that R is a domain.

   (b) Show that there is an algebra homomorphism $R \to K[T]$, $X \mapsto T^2$, $Y \mapsto T^3$, and that this induces an isomorphism $\mathrm{Quot}(R) \cong K(T)$.

   (c) Show that T is integral over R, but that $T \notin R$. In particular, R is not integrally closed.

   (d) Compute the integral closure of R inside $K(T)$.

3. Let R be Noetherian and $R \leq T$ a finitely generated R-algebra. Let $S \leq T$ be a subring containing R. Show that if T is integral over S, then S is finitely generated as an R-algebra.

   Hint. Take a finite set of generators for T as an R-algebra. Each of these is integral over S, so satisfies a monic polynomial. Consider the R-algebra generated by all the coefficients of these polynomials.

4. Let $R \leq S$ be rings. Show that being integral over R is a local property; in other words, prove that the following are equivalent for $x \in S$.

   (a) x is integral over R.

   (b) $x/1$ in $S_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \lhd R$.

   (c) $x/1$ in $S_{\mathfrak{m}}$ is integral over $R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \lhd R$.

   Hint. For (c) $\Rightarrow$ (a), consider the set $I(x) := \{r \in R : rx \text{ integral over } R\}$. Show that $I(x)$ is an ideal in R, and that $I(x)$ is not contained in any maximal ideal.

5. Let $R \leq S$ be rings, $\bar{R}$ the integral closure of R in S, and $\Sigma \subset R$ a multiplicatively closed subset. Prove that $\bar{R}_{\Sigma}$ is the integral closure of R_{Σ} in S_{Σ}.

   Deduce that being integrally closed is a local property; in other words, the following are equivalent for a domain R with quotient field K.

   (a) R is intgerally closed in K.

   (b) $R_{\mathfrak{p}}$ is integrally closed in K for all prime ideals $\mathfrak{p} \lhd R$.

   (c) $R_{\mathfrak{m}}$ is integrally closed in K for all maximal ideals $\mathfrak{m} \lhd R$.

6. Let $R \leq S$ be rings, $\mathfrak{p} \lhd R$ a prime ideal, and $\kappa(\mathfrak{p}) := \mathrm{Quot}(R/\mathfrak{p})$ its residue field. Show that the prime ideals of S lying above $\mathfrak{p}$ are in bijection with the prime ideals of $S \otimes_R \kappa(\mathfrak{p})$. This proves Corollary 12.11.

   Assume now that $R \leq S$ is integral and that S is Noetherian. Deduce that there are only finitely many primes of S lying above $\mathfrak{p}$. Geometrically, this says that integral ring homomorphisms correspond to surjective maps with finite fibres.

   Hint: Show that $\bar{S} := S \otimes_R \kappa(\mathfrak{p})$ is Noetherian, and is integral over $\kappa(\mathfrak{p})$. Show that every prime in $\bar{S}$ is maximal, and use that $\mathrm{nil}(\bar{S})$ is a finite intersection of primes.

Noether Normalisation

7. We need to prove the Noether Normalisation Lemma for finite fields.

   Let K be a finite field, and $R = K[x_1, \dots, x_{n+1}]$ a finitely generated K-algebra. Assume we have $f \neq 0$ in the kernel of $K[X_1, \dots, X_{n+1}] \to R$, $X_i \mapsto x_i$. By renumbering we may assume that $t := x_{n+1}$ occurs in f.

   Prove that for d suffiently large, and setting $Z_r := X_r - T^{d^r}$, the polynomial

   $$g(Z_1, \dots, Z_n, T) := f(Z_1 + T^d, \dots, Z_n + T^{d^n}, T)$$

   is monic in T, and hence that t is integral over $\bar{R} := K[z_1, \dots, z_n]$, where $z_r := x_r - t^{d^r}$.

   Hint. Show that after making the substitution, a monomial $X_1^{a_1} \cdots X_n^{a_n} T^b$ is sent to a polynomial in Z_i and T of the form $T^N +$ lower degree terms. Now show that if $d > \deg(f)$, then the polynomial g is monic.

   The proof now follows by induction on n as in the lectures.

8. Let R be a domain with quotient field K, and $R \leq S$ a finitely generated R-algebra.

   (a) Show that $K \otimes_R S$ is a finitely generated K-algebra.

   (b) Apply the Noether Normalisation Lemma to obtain a subring $K[X_1, \dots, X_n]$ over which $K \otimes_R S$ is finite.

   (c) By examining the proof of the Noether Normalisation Lemma, show that there exists $0 \neq a \in R$ such that S_a is finite over $R_a[X_1, \dots, X_n]$.

Faithful flatness

9. Let $R \leq S$ be rings with S flat as an R-module. Show that the following are equivalent.

   (a) For all primes $\mathfrak{p} \lhd R$, there exists a prime $P \lhd S$ lying over $\mathfrak{p}$.

   (b) For all maximal ideals $\mathfrak{m} \lhd R$, the ideal $\mathfrak{m}S \lhd S$ is proper.

   (c) If M is a non-zero R-module, then $S \otimes_R M$ is a non-zero S-module.

   (d) S is a faithfully flat R-module.

   Hint. For (b) $\Rightarrow$ (c) reduce to the case when M is cyclic, so isomorphic to R/I for some ideal I. For (d) $\Rightarrow$ (a) show that $R/\mathfrak{p} \to S/\mathfrak{p}S$ is injective by applying $S \otimes_R -$ and composing with the natural map $S \otimes_R S/\mathfrak{p}S \to S/\mathfrak{p}S$.

   Recall that every free R-module is faithfully flat. In particular, $R \to R[X]$ is faithfully flat, as is every field extension.

10. Let $R \leq S$ be flat, $P \lhd S$ a prime, and $\mathfrak{p} := P \cap R$. Show that $R_\mathfrak{p} \leq S_P$ is flat. Use the previous exercise to show that $R_\mathfrak{p} \leq S_P$ is faithfully flat. Deduce that $R \leq S$ has the Going-Down Property.