Part I
Basic notions

1 Rings

A ring R is an additive group equipped with a multiplication $R \times R \to R$, $(r,s) \mapsto rs$, which is associative $(rs)t = r(st)$, bilinear $r(s+s') = rs + rs'$ and $(r+r')s = rs + r's$, and unital, so there exists $1 \in R$ with $1r = r = r1$ for all $r \in R$.

Example 1.1. *The ring of integers $\mathbb{Z}$, as well as any field k, are commutative rings, so $rs = sr$ for all elements r, s. Matrices over a field $\mathbb{M}_n(k)$ form a non-commutative ring. If R is a ring, then we have the ring $R[t]$ of polynomials with coefficients in R.*

A ring homomorphism $f\colon R \to S$ is a map of abelian groups which is compatible with the multiplicative structures, so $f(rr') = f(r)f(r')$ and $f(1_R) = 1_S$. The composition of two ring homomorphisms is again a ring homomorphism, composition is asociative, and the identity map on R is a ring homomorphism (so we have a category of rings). We say that f is an isomorphism if there exists a ring homomorphism $g\colon S \to R$ with $gf = \mathrm{id}_R$ and $fg = \mathrm{id}_S$.

1.1 Subrings and ideals

Given a ring R, a subring is an additive subgroup $S \subset R$ containing the identity and which is closed under multiplication, so $1_R \in S$ and $s, s' \in S$ implies $ss' \in S$. Thus S is itself a ring, and the inclusion map $S \rightarrowtail R$ is a ring homomorphism.

A right ideal $I \leq R$ is an additive subgroup which is closed under right multiplication by elements of R, so $x \in I$ and $r \in R$ implies $xr \in I$. Similarly for left ideals and two-sided ideals. Given a two-sided ideal $I \lhd R$, the additive quotient R/I is naturally a ring, via the multiplication $\bar{r}\,\bar{s} := \overline{rs}$, and the natural map $R \to R/I$ is a ring homomorphism.

Given a ring homomorphism $f\colon R \to S$, its kernel $\mathrm{Ker}(f) := \{r \in R : f(r) = 0\}$ is always a two-sided ideal of R, its image $\mathrm{Im}(f)$ is a subring of S, and there is a natural isomorphism $R/\mathrm{Ker}(f) \xrightarrow{\sim} \mathrm{Im}(f)$.

Example 1.2. *The centre $Z(R)$ of a ring R is always a commutative subring*

$$Z(R) := \{r \in R : sr = rs \quad \text{for all } s \in R\}.$$

A division ring R is one where every non-zero element has a (two-sided) inverse, so $r \neq 0$ implies there exists $s \in R$ with $rs = 1 = sr$. A field is the same thing as a commutative division ring.

Let k be a commutative ring. A k-algebra is a ring R together with a fixed ring homomorphism $k \to Z(R)$. Note that every ring is uniquely a $\mathbb{Z}$-algebra. We will mostly be interested in k-algebras when k is a field.

1.2 Idempotents

An idempotent in a ring R is an element $e \in R$ such that $e^2 = e$. Clearly 1_R and 0 are idempotents. If e is an idempotent, then so too is $e' := 1 - e$, and $ee' = 0 = e'e$.

If $e \in R$ is an idempotent, then eRe is again a ring, with the induced multiplication and identity e. Thus eRe is in general not a subring of R.

Given two rings R and S we can form their direct product $R \times S$, which is a ring via the multiplication $(r, s) \cdot (r', s') := (rr', ss')$. Note that this has unit $(1_R, 1_S)$. Moreover, the element $e := (1_R, 0)$ is a central idempotent, and $R \cong e(R\times S)e$. Conversely, if $e \in R$ is a central idempotent, then R is isomorphic to the direct product $eRe \times e'Re'$.

We say that R is indecomposable if it is not isomorphic to a direct product, equivalently if it has no non-trivial central idempotents.

1.3 Local rings

A ring R is called local if the set of non-invertible elements of R forms an ideal $I \lhd R$, equivalently if R has a unique maximal right (or left) ideal. In this case R/I is necessarily a division ring. If R is local, then it has no non-trivial idempotents. (If $e \neq 0, 1$ is an idempotent, then since $ee' = 0$, both e and e' are non-invertible, and hence their sum $e+e' = 1$ is non-invertible, a contradiction.)

A discrete valuation ring, or DVR, is a commutative local domain R containing a non-zero element t such that every ideal is of the form t^nR. (Recall that R is a domain if it has no proper zero divisors, so $rs = 0$ implies $r = 0$ or $s = 0$.)

One example of a DVR is the ring of formal power series $k[\![t]\!]$, where k is a field. This has elements $\sum_{n\geq 0} a_nt^n$ with $a_n \in k$, and the obvious addition

$$\sum_n a_nt^n + \sum_n b_nt^n := \sum_n (a_n + b_n)t^n$$

and multiplication

$$\Big(\sum_n a_nt^n\Big)\Big(\sum_n b_nt^n\Big) := \sum_n \Big(\sum_{i+j=n} a_ib_j\Big)t^n.$$

The maximal ideal is precisely those elements $\sum_n a_nt^n$ with $a_0 = 0$.

More generally, a principal ideal domain, or PID, is a commutative domain R such that every ideal is principal, so of the form rR for some $r \in R$. The basic example of a PID is the ring of integers $\mathbb{Z}$, and many results extend from $\mathbb{Z}$ to every PID. For example, we say that $p \in R$ is prime provided R/pR is a field, we say r divides s, written $r|s$, provided $sR \leq rR$.

Then every non-zero element $r \in R$ can be written essentially uniquely as a product of primes, so $r = up_1^{m_1} \cdots p_r^{m_r}$ with p_i distinct primes (so $p_iR \neq p_jR$ for $i \neq j$), $m_i \geq 1$ and $u \in R$ invertible.

Let R be a PID. Since R is a domain, we can also form its field of fractions K, which is the smallest field into which R embeds. Given a prime $p \in R$, we can then consider the following subring of K

$$R_p := \{a \in K : ar \in R \text{ for some } r \in R, \text{ not divisible by } p\}.$$

It is the largest subring of K containing R for which p is not invertible, and is therefore a DVR with maximal ideal generated by p and quotient field $R_p/pR_p \cong R/pR$.

We also have the notion of a non-commutative DVR, which is a domain R containing a non-zero element t such that $Rt = tR$ and every right ideal is of the form $t^n R$. Note that we then obtain a ring automorphism $r \mapsto \bar{r}$ of R such that $tr = \bar{r}t$.

2 Modules

Let R be a ring. A (right) R-module is an additive group M together with a map $M \times R \to M$, $(m, r) \mapsto mr$, which is associative $(mr)s = m(rs)$, bilinear $m(r + r') = mr + mr'$ and $(m + m')r = mr + m'r$, and unital $m1 = m$. It follows from these that also $m0 = 0$ and $m(-1) = -m$.

Example 2.1. *Modules over a field k are the same as k-vector spaces. Modules over the integers $\mathbb{Z}$ are the same as abelian groups.*

A module homomorphism $f\colon M \to N$ is a map of abelian groups which is compatible with the R-actions, so $f(mr) = f(m)r$. Composition of homomorphisms is again a homomorphism, composition is associative, and the identity on M is a homomorphism (so we have a category of R-modules). We say that f is an isomorphism if there exists a homomorphism $g\colon N \to M$ with $gf = \mathrm{id}_M$ and $fg = \mathrm{id}_N$.

We write $\mathrm{Hom}_R(M, N)$ for the set of all R-module homomorphisms $f\colon M \to N$. Note that $\mathrm{Hom}_R(M, N)$ is again an additive group, via $(f + f')(m) := f(m) + f'(m)$. Moreover, composition is bilinear $(f + f')g = fg + f'g$ and $f(g + g') = fg + fg'$ (so we have a preadditive category of R-modules.)

As a special case, an R-homomorphism from M to itself is called an R-endomorphism, and we write $\mathrm{End}_R(M)$. This has the structure of a ring, via

$$(fg)(m) := f(g(m)),$$

with unit the identity map. An isomorphism from M to itself is called an automorphism; these form the units in $\mathrm{End}_R(M)$.

Finally, we observe that $\mathrm{Hom}_R(M, N)$ is a right $\mathrm{End}_R(M)$-module, and also a left $\mathrm{End}_R(N)$-module. Moreover, associativity of composition gives $\nu(f\mu) = (\nu f)\mu$ for all $\mu \in \mathrm{End}_R(M)$, $\nu \in \mathrm{End}_R(N)$ and $f \in \mathrm{Hom}_R(M, N)$. Thus $\mathrm{Hom}_R(M, N)$ is an $\mathrm{End}_R(N)$-$\mathrm{End}_R(M)$-bimodule.

2.1 Submodules and quotient modules

Let M be an R-module. A submodule $U \leq M$ is an additive subgroup, closed under the action of R, so $ur \in U$ for all $u \in U$ and $r \in R$. Thus U is itself an R-module, and the inclusion map $U \rightarrowtail M$ is a module homomorphism.

Given a submodule $U \leq M$, the additive quotient M/U is naturally an R-module, via the action $\bar{m}\, r := \overline{mr}$, and the natural map $M \twoheadrightarrow M/U$ is a module homomorphism.

Let $f\colon M \to N$ be a module homomorphism. Then its kernel $\mathrm{Ker}(f) := \{m \in M : f(m) = 0\}$ is a submodule of M, its image $\mathrm{Im}(f) := \{f(m) : m \in M\}$ is a submodule of N, and there is a natural isomorphism $M/\operatorname{Ker}(f) \xrightarrow{\sim} \mathrm{Im}(f)$. We set $\mathrm{Coker}(f) := N/\operatorname{Im}(f)$, called the cokernel of f.

Example 2.2. *The ring R is naturally a right (and left) module over itself, called the regular module. The submodules are precisely the right (or left) ideals.*

Let $f\colon M \to N$ be a module homomorphism. We say that f is a monomorphism if it is injective, equivalently $\mathrm{Ker}(f) = 0$. We say that f is an epimorphism if it is surjective, equivalently $\mathrm{Coker}(f) = 0$. Finally, f is an isomorphism if and only if it is bijective.

2.2 k-algebras

Let k be a commutative ring, and M and N two k-modules. Then the abelian group $\mathrm{Hom}_k(M,N)$ is naturally a k-module, via $(fa)(m) := f(m)a$ for $f\colon M \to N$, $a \in k$ and $m \in M$. Similarly, $\mathrm{End}_k(M)$ is naturally a k-algebra. Note that the k-action on $\mathrm{Hom}_k(M,N)$ is induced by either the action of $\mathrm{End}_k(M)$ or the action of $\mathrm{End}_k(N)$.

Example 2.3. *Let k be a field. Then a k-module is a vector space, and the set of linear maps $\mathrm{Hom}_k(k^m,k^n)$ is also a vector space, isomorphic to matrices $\mathbb{M}_{n\times m}(k)$. In particular, $\mathrm{End}_k(M)$ is isomorphic to the matrix algebra $\mathbb{M}_m(k)$.*

A k-algebra is a ring together with a ring homomorphism $k \to Z(R)$ from k to the centre of R. In this case every R-module has the induced structure of a k-module, and every R-module homomorphism is necessarily a k-module homomorphism. Thus $\mathrm{Hom}_R(M,N)$ is a k-submodule of $\mathrm{Hom}_k(M,N)$, and similarly $\mathrm{End}_R(M)$ is a k-subalgebra of $\mathrm{End}_k(M)$.

Example 2.4. *Let k be a field and consider the polynomial ring $R = k[t]$. Then giving an R-module is equivalent to giving a vector space M together with a linear endomorphism $T \in \mathrm{End}_k(M)$ describing the action of t, so $mt := T(m)$.*

From now on we will usually work in this relative setting, so R will always be a k-algebra.

2.3 Simple modules

A non-zero module M is called simple provided it has no submodules other than 0 or M.

Lemma 2.5 (Schur's Lemma)**.** *Let S be simple. Then every non-zero $f\colon S \to M$ is injective, and every non-zero $g\colon N \to S$ is surjective. In particular, every non-zero endomorphism of S is an automorphism, so $\mathrm{End}_R(M)$ is a division ring.*

Proof. Given $f\colon S \to M$ we have $\mathrm{Ker}(f) \leq S$, so either $\mathrm{Ker}(f) = 0$ and f is injective, or else $\mathrm{Ker}(f) = S$ and f is zero. Similarly for $g\colon N \to S$, using $\mathrm{Im}(g) \leq N$. □

A composition series of a module M is a chain of submodules

$$0 = M_n \leq \cdots \leq M_1 \leq M_0 = M$$

such that each successive subquotient M_i/M_{i+1} is simple. The series has length n if there are n such simple subquotients.

Theorem 2.6 (Jordan–Hölder Theorem)**.** *If M has a composition series of length n, then any other composition series also has length n, so we may write $\ell(M) = n$.*

Moreover, the isomorphism classes of the simple subquotients which occur, together with their multiplicities, are independent of the choice of composition series.

Proof. Let $M' \leq M$ be a maximal submodule, so $S := M/M'$ is simple. Let $M_\bullet$ be any composition series of M, say of length n. Then the map $f_i \colon M_i \to M \to M/M'$ is either zero or surjective. Since $M_n = 0$, the map $f_n = 0$, whereas $M_0 = M$, so f_0 is surjective. Let j be maximal such that f_j is surjective. We set $N_i := \mathrm{Ker}(f_i)$, so $N_i = M_i$ for $i > j$. We claim that there is a composition series

$$0 = N_n \leq N_{n-1} \leq \cdots \leq N_{j+1} = N_j \leq \cdots \leq N_1 \leq N_0 = M'$$

and moreover that $N_i/N_{i+1} \cong M_i/M_{i+1}$ for $i < j$.

To see this, take $i < j$. By construction the map f_{i+1} is onto and factors through f_i, so $M_{i+1} \rightarrowtail M_i \twoheadrightarrow M_i/N_i \cong S$. We thus have the induced isomorphism $M_{i+1}/N_{i+1} \xrightarrow{\sim} M_i/N_i$, and hence

$$M_{i+1} \cap N_i = N_{i+1} \quad \text{and} \quad M_{i+1} + N_i = M_i.$$

It follows that the composition $N_i \rightarrowtail M_i \twoheadrightarrow M_i/M_{i+1}$ is also onto, so yields an isomorphism $N_i/N_{i+1} \xrightarrow{\sim} M_i/M_{i+1}$.

We have therefore constructed a composition series $M'_\bullet$ for M' of length $n-1$, and the simple subquotients for $M'_\bullet$ are those for $M_\bullet$ except $M_j/M_{j+1} \cong M/M'$. The result for M now follows by induction from the result for M'. □

Example 2.7. *Let k be a field. Then the only simple module is k itself, and the length of a vector space is just its dimension. If R is a k-algebra, then necessarily every R-module which is finite dimensional over k is of finite length.*

Example 2.8. *Let R be a principal ideal domain, for example the ring of integers $\mathbb{Z}$ or the polynomial ring $k[t]$ over a field k. Then the simple R-modules are all of the form R/pR, where $p \in R$ is a prime. More generally, the R-module R/p^nR has length n with all composition factors isomorphic to R/pR. On the other hand, R itself has no simple submodules.*

2.4 Direct sums and products

Given a family of R-modules M_i indexed by a set I, we can form their direct product $\prod_i M_i$ by taking as elements all tuples (m_i) with $m_i \in M_i$, and pointwise module operations, so

$$(m_i)r + (n_i) := (m_i r + n_i).$$

We can also form their direct sum $\coprod_i M_i$, or $\bigoplus_i M_i$, as the submodule of $\prod_i M_i$ consisting of those elements (m_i) of finite support, so $m_i = 0$ for all but finitely many $i \in I$. If I is a finite set, then clearly $\coprod_i M_i = \prod_i M_i$.

There are natural homomorphisms $\pi_i \colon \prod_i M_i \twoheadrightarrow M_i$ and $\iota_i \colon M_i \rightarrowtail \coprod_i M_i$ such that $\pi_i \iota_i = \mathrm{id}_{M_i}$ and $\pi_j \iota_i = 0$ for $i \neq j$.

If $M_i \cong M$ for all $i \in I$, then we also write $M^{(I)}$ for the direct sum, and M^I for the direct product.

Lemma 2.9. *Let L, M and N be R-modules. Then $M \cong L \oplus N$ if and only if there exist homomorphisms*

$$L \xrightarrow{i_L} M \xrightarrow{p_L} L \quad \textit{and} \quad N \xrightarrow{i_N} M \xrightarrow{p_N} N$$

such that

$$\mathrm{id}_L = p_L i_L, \quad \mathrm{id}_N = p_N i_N, \quad \mathrm{id}_M = i_L p_L + i_N p_N.$$

Proof. Note first that the direct sum $L \oplus N$ comes equipped with the canonical maps $\iota_L, \pi_L, \iota_N, \pi_N$, and these obviously satisfy the three conditions.

Suppose first that $f\colon M \xrightarrow{\sim} L \oplus N$ is an isomorphism. Then we can take as the four maps $f^{-1}\iota_L, \pi_L f, f^{-1}\iota_N, \pi_N f$.

Conversely, suppose we have the four maps. Note that $p_L i_N = 0$ and $p_N i_L = 0$. For example,

$$p_L i_N = p_L(i_L p_L + i_N p_N) i_N = p_L i_N + p_L i_N = 2 p_L i_N,$$

so $p_L i_N = 0$. We can now define $f\colon M \to L \oplus N$ and $g\colon L \oplus N \to M$ via

$$f := \iota_L p_L + \iota_N p_N \quad \text{and} \quad g := i_L \pi_L + i_N \pi_N,$$

and check that $gf = \mathrm{id}_M$ and $fg = \mathrm{id}_{L\oplus N}$. □

As a special case we see that if $L, N \leq M$ are submodules, then $M \cong L \oplus N$ whenever $L \cap N = 0$ and $L + N = M$. For, we take i_L to be the inclusion map, whereas to define p_L we observe that the composition $L \rightarrowtail M \twoheadrightarrow M/N$ is an isomorphism, so we set p_L to be the induced map $M \twoheadrightarrow M/N \xrightarrow{\sim} L$. Similarly for i_N and p_N.

2.5 Indecomposables and Fitting's Lemma

A module M is indecomposable if for any decomposition $M \cong L \oplus N$ we must have $L = 0$ or $N = 0$.

Lemma 2.10 (Fitting's Lemma). *Suppose $\mathrm{End}_R(M)$ is a local ring. Then M is indecomposable. The converse holds whenever M has finite length, in which case every non-invertible endomorphism is nilpotent.*

Proof. Given $M \cong L \oplus N$ we have the corresponding element $e_L := i_L p_L \in \mathrm{End}_R(M)$. Then $e_L^2 = e_L$, so e_L is an idempotent. Now, if $\mathrm{End}_R(M)$ is a local ring, then it has no idempotent elements other than id_M and 0, so either $e_L = 0$ and $L = 0$, or $e_L = \mathrm{id}_M$ and $N = 0$. Thus M is indecomposable.

Conversely, suppose that M has finite length and let $f \in \mathrm{End}_R(M)$. Then we have a chains of submodules

$$M \geq \mathrm{Im}(f) \geq \mathrm{Im}(f^2) \geq \cdots \quad \text{and} \quad 0 \leq \mathrm{Ker}(f) \leq \mathrm{Ker}(f^2) \leq \cdots$$

and since M has finite length, both of these must stabilise. Thus there exists n such that both $\mathrm{Im}(f^n) = \mathrm{Im}(f^{n+1})$ and $\mathrm{Ker}(f^n) = \mathrm{Ker}(f^{n+1})$.

It follows that $M \cong \mathrm{Im}(f^n) \oplus \mathrm{Ker}(f^n)$. For, take $x \in \mathrm{Im}(f^n) \cap \mathrm{Ker}(f^n)$. Then $x = f^n(y)$, and $0 = f^n(x) = f^{2n}(y)$. Thus $y \in \mathrm{Ker}(f^{2n}) = \mathrm{Ker}(f^n)$, so $x = f^n(y) = 0$. Similarly, given $m \in M$, we have $f^n(m) = f^{2n}(m')$, and so $m = x + y$ where $x = f^n(m') \in \mathrm{Im}(f^n)$ and $y = m - x \in \mathrm{Ker}(f^n)$.

In particular, if M is indecomposable of finite length, then every endomorphism f is either nilpotent ($\mathrm{Im}(f^n) = 0$, so $f^n = 0$) or an automorphism ($\mathrm{Im}(f^n) = M$ and $\mathrm{Ker}(f^n) = 0$).

Finally, we show that the nilpotent endomorphisms form a two-sided ideal, so $\mathrm{End}_R(M)$ is local.

To see this note first that a product fg is invertible if and only if both f and g are invertible. For, if fg is invertible, then f is surjective and g is injective, so neither is nilpotent, so both are invertible; the converse is clear. Next, if f is nilpotent, then $1 + f$ is invertible, with inverse $1 - f + f^2 - f^3 + \cdots$. Finally, suppose f is nilpotent and $\phi := f + g$ is invertible. Then $g = \phi - f = \phi(1 - \phi^{-1} f)$ is invertible. □

3 Modules for Principal Ideal Domains

Let R be a (commutative) principal ideal domain, so every ideal is of the form xR for some $x \in R$. Examples include $\mathbb{Z}$ and $k[t]$ for a field k.

We write $x|y$ provided $yR \subset xR$. Also, given any $x, y \in R$ we have their greatest common divisor d and lowest common multiple m, defined via

$$xR + yR = dR \quad \text{and} \quad xR \cap yR = mR.$$

Note also that R is Noetherian; that is, every ascending chain of ideals

$$I_1 \subset I_2 \subset \cdots$$

necessarily stabilises. For, the union $I := \bigcup_i I_i$ is again an ideal, so of the form xR, and if $x \in I_N$, then $I_n = I_N = I$ for all $n \geq N$.

A vector $(r_1, \dots, r_n) \in R^n$ is called unimodular provided the r_i generate the unit ideal, so $\sum_i r_i R = R$. We write $\mathrm{GL}_n(R)$ for the invertible matrices in $\mathbb{M}_n(R)$, so those whose determinant is a unit in R. We begin with a nice lemma about extending unimodular vectors to invertible matrices.

Lemma 3.1. *Every unimodular vector appears as the first row of an invertible matrix.*

Proof. We prove this by induction on n. The case $n = 1$ being trivial, since r_1 is necessarily a unit.

Consider a unimodular vector $(a, r_1, \dots, r_n)$. Let d be the greatest common divisor of the r_i, and write $r_i = ds_i$. Then $(s_1, \dots, s_n)$ is unimodular, so by induction there is a matrix $M \in \mathrm{GL}_n(R)$ having first row $(s_1, \dots, s_n)$. Next, a and d generate the unit ideal, so we can write $ax + dy = 1$. We now consider the matrix $\tilde{M} \in \mathbb{M}_{n+1}(R)$ having as first row $(a, r_1, \dots, r_n)$, as i-th row $(0, m_{i1}, \dots, m_{in})$ for $1 < i \leq n$, and as last row $(y, s_1 x, \dots, s_n x)$. Finally we compute the determinant of $\tilde{M}$ by expanding along the first column. We get

$$\det \tilde{M} = (-1)^{n-1}(ax + dy) \det M = (-1)^{n-1} \det M,$$

so $\det \tilde{M}$ is a unit and $\tilde{M} \in \mathrm{GL}_{n+1}(R)$. $\square$

Example 3.2. *Consider the unimodular vector* $(6, 10, 15) \in \mathbb{Z}^3$. *We have* $\gcd(10, 15) = 5$, *yielding the unimodular vector* $(2, 3)$ *which we can complete to an invertible matrix* $M = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. *Now* $6 - 5 = 1$, *so* $x = 1$ *and* $y = -1$, *so the proof of the lemma gives the invertible matrix*

$$\tilde{M} = \begin{pmatrix} 6 & 10 & 15 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix}.$$

We can now give the Smith Normal Form for matrices over principal ideal domains.

Theorem 3.3 (Smith Normal Form)**.** *Let* $A \in \mathbb{M}_{m \times n}(R)$ *be any matrix. Then* A *is equivalent to a matrix in block form*

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \mathrm{diag}(d_1, \dots, d_r), \quad d_1 | d_2 | \cdots | d_r.$$

Recall that equivalence of matrices is generated by row and column operations, which is the same as acting on left by matrices in $\mathrm{GL}_m(R)$ and on the right by matrices in $\mathrm{GL}_n(R)$.

Proof. We start by noting the following consequences of the lemma.

(1) Let A be any matrix, say with first row $(r_1, \dots, r_n)$. Let $I = aR$ be the ideal generated by the r_i, so $r_i = as_i$ and $(s_1, \dots, s_n)$ is unimodular. By the lemma there exists a matrix $M \in \mathrm{GL}_n(R)$ with first row $(s_1, \dots, s_n)$, and so AM^{-1} has first row $(a, 0, \dots, 0)$.

(2) Similarly if $J = bR$ is the ideal generated by the entries in the first column of A, then there exists $N \in \mathrm{GL}_m(R)$ such that $N^{-1}A$ has first column $(b, 0, \dots, 0)^t$.

Now, starting from our matrix A, we repeatedly apply these constructions. This yields an ascending chain of ideals

$$I_1 \subset J_1 \subset I_2 \subset J_2 \subset \cdots$$

which must then stabilise since R is Noetherian. It follows that A is equivalent to a matrix of the form

$$\begin{pmatrix} e_1 & 0 \\ 0 & A' \end{pmatrix}.$$

By induction we know that A' is equivalent to a matrix having the required form, and so we have reduced to the case when our matrix is of the form

$$\begin{pmatrix} D' & 0 \\ 0 & 0 \end{pmatrix}, \quad D' = \mathrm{diag}(e_1, d_1', \dots, d_r'), \quad d_1' | d_2' | \cdots | d_r'.$$

Now consider the matrix $\begin{pmatrix} e_1 & 0 \\ 0 & d_1' \end{pmatrix}$. This is equivalent to $\begin{pmatrix} e_1 & d_1' \\ 0 & d_1' \end{pmatrix}$, so we can apply our reduction process to obtain an equivalent matrix $\begin{pmatrix} d_1 & 0 \\ 0 & e_2 \end{pmatrix}$, where $d_1 = \gcd(e_1, d_1')$, and then necessarily $e_2 = \mathrm{lcm}(e_1, d_1')$ (compare determinants).

We now continue in this way, using the matrix $\begin{pmatrix} e_2 & 0 \\ 0 & d_2' \end{pmatrix}$ and so on, finally yielding a matrix $D = \mathrm{diag}(d_1, \dots, d_{r+1})$, equivalent to D' and satisfying $d_1 | d_2 | \cdots | d_{r+1}$. $\square$

Theorem 3.4 (Structure Theorem for Finitely Generated Modules)**.** *Let M be an indecomposable, finitely generated R-module. Then M is isomorphic to either R itself, or to some R/p^mR where $p \in R$ is prime and $m \geq 0$.*

Moreover, every finitely generated module is isomorphic to a direct sum of indecomposable modules in an essentially unique way. In other words, given

$$M_1 \oplus \cdots \oplus M_r \cong N_1 \oplus \cdots \oplus N_s$$

with M_i and N_j finitely generated and indecomposable, then $r = s$ and, after reordering, $M_i \cong N_i$.

Proof. Since R is Noetherian, any finitely generated module is the cokernel of some map $R^m \to R^n$. Next any such map can be put into Smith Normal Form, and so the cokernel is isomorphic to some $R^a \oplus (R/d_1R) \oplus \cdots \oplus (R/d_rR)$.

Now, writing $d = p_1^{m_1} \cdots p_r^{m_r}$ as a product of primes, we can use the Chinese Remainder Theorem to get

$$R/dR \cong (R/p_1^{m_1}R) \oplus \cdots \oplus (R/p_r^{m_r}R).$$

Thus every finitely generated module is isomorphic to a direct sum of modules of the form R or R/p^mR with $p \in R$ prime and $m \geq 1$.

We need to show that each of these summands is indecomposable. For R itself, this follows from the fact that it is a domain. For, if $R = M \oplus N$, then M and N are ideals in R, say mR and nR, respectively. Now $mn \in M \cap N = 0$, so either $m = 0$ or $n = 0$.

For R/p^mR we have that $\mathrm{End}_R(R/p^mR) \cong R/p^mR$. This is a local ring, with unique maximal ideal pR/p^mR, so the module is indecomposable by Fitting's Lemma.

Finally, we need to show uniqueness. Suppose we have a finitely generated module

$$M \cong R^n \oplus \bigoplus_{p,i} (R/p^{m_{p,i}}R).$$

Let K be the field of fractions of R. Then $R \otimes_R K \cong K$ whereas $(R/p^rR) \otimes_R K = 0$. Thus $n = \dim_K(M \otimes_R K)$.

Now let $p \in R$ be a fixed prime, so that $K(p) := R/pR$ is a field. Consider the R-submodule Mp^s of M, and the corresponding $K(p)$-vector space $Mp^s \otimes_R K(p)$. We note that

$$Rp^s \otimes_R K(p) \cong K(p) \quad \text{and} \quad (R/q^mR)p^s \otimes_R K(p) = 0 \text{ for } qR \neq pR.$$

Also, $(R/p^mR)p^s \otimes_R K(p)$ equals $K(p)$ if $m > s$, and is zero if $m \leq s$. Thus we can compute the sizes of the sets $\{i : m_{p,i} \geq s\}$ for all s, and hence compute the numbers $m_{p,i}$ themselves, just by computing the dimensions of the $K(p)$-vector spaces $Mp^s \otimes_R K(p)$. □

Example 3.5. *Let $R = \mathbb{Z}$. Then this result shows that the indecomposable finite abelian groups are the cyclic groups of prime power order $\mathbb{Z}/(p^n)$, and every finite abelian group is isomorphic to a finite direct sum of indecomposable ones, in an essentially unique way.*

Example 3.6. *Let $R = k[t]$ where k is an algebraically-closed field. The primes in $k[t]$ are the polynomials $t - \lambda$ for $\lambda \in k$. Thus, choosing the basis $\{1, t, t^2, \ldots, t^{n-1}\}$ for $k[t]/((t-\lambda)^n)$, we see that this indecomposable module has underlying vector space k^n, and that t acts via the Jordan block matrix $J_n(\lambda)$ of size n and eigenvalue λ.*

Thus, if we identify $k[t]$-modules with pairs (V, T) consisting of a vector space V together with a linear endomorphism $T \in \mathrm{End}_k(V)$, then this result shows that every square matrix is conjugate to a matrix in Jordan Normal Form.

Example 3.7. *More generally, let $R = k[t]$ for an arbitrary field k. Then the primes correspond to monic irreducible polynomials, and choosing an appropriate basis we can write every matrix in rational canonical form.*

In summary, the finite length indecomposable modules come in families indexed by primes, and each family has a unique indecomposable of length n for each $n \geq 1$. This lies at the heart of understanding what happens for any finite dimensional hereditary (non-commutative) algebra.

4 Short exact sequences

A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is said to be exact if $\mathrm{Im}(f) = \mathrm{Ker}(g)$ as submodules of M. A diagram

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a short exact sequence provided it is exact at L, M and N. This is equivalent to saying f is a monomorphism, g is an epimorphism, and $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Two short exact sequences fitting into a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f'} & M' & \xrightarrow{g'} & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mu} & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

are said to be equivalent. Note that in an equivalence of short exact sequences, the homomorphism μ is necessarily an isomorphism. (This is either an easy diagram chase, or a consequence of the more general Snake Lemma below.)

A split short exact sequence is one equivalent to the trivial sequence

$$0 \longrightarrow L \xrightarrow{\iota_L} L \oplus N \xrightarrow{\pi_N} N \longrightarrow 0 \ .$$

Lemma 4.1. *The following are equivalent for a short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \ .$$

1. *There exists a retract $r \colon M \to L$, so $rf = \mathrm{id}_L$.*
2. *There exists a section $s \colon N \to M$, so $gs = \mathrm{id}_N$.*
3. *The sequence is split.*

Proof. Given a retract r, we have $M = \mathrm{Ker}(r) \oplus \mathrm{Im}(f)$. Setting $\mu := \iota_L r + \pi_N g \colon M \to L \oplus N$ we see that the sequence is split. Similarly if we have a section s. Conversely, if the sequence is split, then we have an isomorphism $\mu \colon M \xrightarrow{\sim} L \oplus N$, so we have the retract $r := \pi_L \mu$ and the section $s := \mu^{-1} \iota_N$. $\square$

4.1 Snake Lemma

Lemma 4.2 (Snake Lemma)**.** *Consider a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} & & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & & \end{array}$$

Then we obtain an exact sequence

$$\mathrm{Ker}(\lambda) \xrightarrow{f} \mathrm{Ker}(\mu) \xrightarrow{g} \mathrm{Ker}(\nu) \xrightarrow{\delta} \mathrm{Coker}(\lambda) \xrightarrow{f'} \mathrm{Coker}(\mu) \xrightarrow{g'} \mathrm{Coker}(\nu).$$

The connecting homomorphism δ satisfies $\delta(n) = l' + \mathrm{Im}(\lambda)$ whenever there exists $m \in M$ such that both $n = g(m)$ and $f'(l') = \mu(m)$.

Moreover, if f is injective, then so too is $\mathrm{Ker}(\lambda) \to \mathrm{Ker}(\mu)$, *and similarly if g' is surjective, then so too is* $\mathrm{Coker}(\mu) \to \mathrm{Coker}(\nu)$.

Proof. Some diagram chasing.

It is clear that f restricts to a map $\mathrm{Ker}(\lambda) \to \mathrm{Ker}(\mu)$. For, if $l \in \mathrm{Ker}(\lambda)$, then $\mu f(l) = f'\lambda(l) = 0$, so $f(l) \in \mathrm{Ker}(\mu)$. Moreover, this restriction is an R-module homomorphism, and it is injective whenever f is injective. Similarly g restricts to an R-module homomorphism $\mathrm{Ker}(\mu) \to \mathrm{Ker}(\nu)$, and $gf = 0$. To check exactness at $\mathrm{Ker}(\mu)$, suppose $m \in \mathrm{Ker}(\mu) \cap \mathrm{Ker}(g)$. Then $m = f(l)$, and $f'\lambda(l) = \mu f(l) = \mu(m) = 0$, so f' injective implies $\lambda(l) = 0$, so $l \in \mathrm{Ker}(\lambda)$.

We next show that δ is well-defined. Given $n \in \mathrm{Ker}(\nu)$, suppose we have two pairs (l'_1, m_1) and (l'_2, m_2) satisfying the conditions. We set $l' := l'_1 - l'_2$ and $m := m_1 - m_2$. Then $g(m) = 0$, so $m = f(l)$ for some $l \in L$, and also $f'(l') = \mu(m) = \mu f(l) = f'(\lambda(l))$. Since f' is injective we get $l' = \lambda(l) \in \mathrm{Im}(\lambda)$, and hence that $l'_1 + \mathrm{Im}(\lambda) = l'_2 + \mathrm{Im}(\lambda)$. Similarly reasoning shows that δ is an R-module homomorphism.

For the exactness at $\mathrm{Ker}(\nu)$, suppose first that $n = g(m)$ for some $m \in \mathrm{Ker}(\nu)$. Then we can take $l' = 0$, and so $\delta(n) = 0$. Conversely, suppose $\delta(n) = 0$, and take some pair (l', m). Then $l' = \lambda(l) \in \mathrm{Im}(\lambda)$, so $\mu(m) = f'(l') = f'\lambda(l) = \mu f(l)$. Thus we could also have taken the pair $(0, m - f(l))$. In particular, $m - f(l) \in \mathrm{Ker}(\mu)$ and $n = g(m - f(l))$.

The remaining parts are all dual. $\square$

4.2 Kernels and Cokernels revisited

Let $f\colon L \to M$ be a map of R-modules. The definition of the kernel of f as being a certain subset of L is in practice too rigid; we need a definition that allows invariance up to isomorphism.

A kernel for f consists of a map $i\colon K \to L$ such that $fi = 0$, and with the property that given any map $\alpha\colon X \to L$ such that $f\alpha = 0$, there exists a unique map $\bar{\alpha}\colon X \to K$ with $i\bar{\alpha} = \alpha$.

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\exists!\bar{\alpha}} & \downarrow{\scriptstyle\alpha} & \searrow^{0} & \\ K & \xrightarrow{i} & L & \xrightarrow{f} & M \end{array}$$

Lemma 4.3. *Let $i\colon K \to L$ and $f\colon L \to M$ be maps of R-modules. Then the following are equivalent.*

1. *The map i is a kernel for f.*
2. *The sequence $0 \to K \xrightarrow{i} L \xrightarrow{f} M$ is exact.*
3. *For all R-modules X the sequence*

$$0 \longrightarrow \mathrm{Hom}_R(X, K) \xrightarrow{i_*} \mathrm{Hom}_R(X, L) \xrightarrow{f_*} \mathrm{Hom}_R(X, M).$$

is exact (as right $\mathrm{End}_R(X)$-modules).

Proof. Note that $f_*(\alpha) = f\alpha$, and similarly for i_*.

$1 \Rightarrow 3$. We have $fi = 0$, so $f_* i_* = 0$. The sequence in 3 is then exact provided for all $\alpha\colon X \to L$, there exists a unique $\bar{\alpha}\colon X \to K$ such that $\alpha = i\bar{\alpha}$, which is precisely the universal property in the definition for a kernel.

$2 \Rightarrow 1$. Take $\alpha\colon X \to L$ with $f\alpha = 0$. Then $\mathrm{Im}(\alpha) \leq \mathrm{Ker}(f) = \mathrm{Im}(i)$, and since i is injective there exists a unique (set-theoretic) map $\bar{\alpha}\colon X \to K$ with $\alpha = i\bar{\alpha}$. It is now easy to check that $\bar{\alpha}$ is an R-module homomorphism.

$3 \Rightarrow 2$. Taking $X = K$, we have $fi = f_* i_*(\mathrm{id}_K) = 0$. Taking $X = \mathrm{Ker}(i)$, then we have the inclusion $\beta\colon X \to I$ and $i_*(\beta) = 0$. By uniqueness $\beta = 0$, and hence $\mathrm{Ker}(i) = 0$. Finally, take $X = \mathrm{Ker}(f)$ and $\alpha\colon X \to L$ the inclusion. Since $f_*(\alpha) = 0$ we know that $\alpha = i\bar{\alpha}$ for some $\bar{\alpha}\colon X \to K$. Thus $\mathrm{Ker}(f) = \mathrm{Im}(\alpha) \subset \mathrm{Im}(i)$. □

Dually, a cokernel for f consists of a map $p\colon M \to C$ such that $pf = 0$, and with the property that given any map $\beta\colon M \to X$ such that $\beta f = 0$, there exists a unique map $\bar{\beta}\colon C \to X$ with $\bar{\beta}p = \beta$.

$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{p} & C \\ & \searrow^{0} & \downarrow{\scriptstyle\beta} & \swarrow_{\exists!\bar{\beta}} & \\ & & X & & \end{array}$$

Lemma 4.4. *Let $f\colon L \to M$ and $p\colon M \to C$ be maps of R-modules. Then the following are equivalent.*

1. *The map p is a cokernel for f.*
2. *The sequence $L \xrightarrow{f} M \xrightarrow{p} C \to 0$ is exact.*
3. *For all R-modules X the sequence*

$$0 \longrightarrow \mathrm{Hom}_R(C,X) \xrightarrow{p^*} \mathrm{Hom}_R(M,X) \xrightarrow{f^*} \mathrm{Hom}_R(L,X).$$

   is exact (as left $\mathrm{End}_R(X)$-modules).

Note that the third conditionsin the lemmas are referred to as saying that $\mathrm{Hom}_R(X,-)$ and $\mathrm{Hom}_R(-,X)$ are left exact functors (the latter being contravariant). Also, exactness in the second lemma at $\mathrm{Hom}_R(M,X)$ is often called the Factor Lemma.

It follows from these definitions that kernels, and similarly cokernels, are unique up to unique isomorphism.

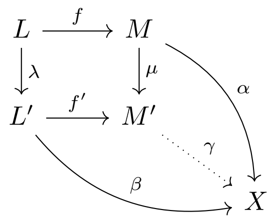
4.3 Push-outs

Given a pair of homomorphisms $f\colon L \to M$ and $\lambda\colon L \to L'$, their push-out consists of a commutative square

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} \\ L' & \xrightarrow{f'} & M' \end{array}$$

with the property that given any pair of homomorphisms $\alpha\colon M \to X$ and $\beta\colon L' \to X$ such that $\alpha\mu = \beta\lambda$, there exists a unique homomorphism $\gamma\colon M' \to$

X with $\alpha = \gamma\mu$ and $\beta = \gamma f'$.



Equivalently we can say that for all X there is an exact sequence

$$0 \to \mathrm{Hom}_R(M', X) \to \mathrm{Hom}_R(M \oplus L', X) \to \mathrm{Hom}_R(L, X),$$

given by composing with $(\mu, f')\colon L' \oplus M \to M$ and $\binom{-f}{\lambda}\colon L \to M \oplus L'$, or alternatively there is an exact sequence

$$L \xrightarrow{\binom{-f}{\lambda}} M \oplus L' \xrightarrow{(\mu, f')} M' \longrightarrow 0.$$

Thus a push-out is the same as a cokernel of the map $\binom{-f}{\lambda}\colon L \to M \oplus L'$, so they exist and are unique up to unique isomorphism.

In a push-out square, parallel maps have naturally isomorphic cokernels.

Lemma 4.5. *Given a push-out square*

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} \\ L' & \xrightarrow{f'} & M' \end{array}$$

we have natural isomorphisms $\mathrm{Coker}(f) \cong \mathrm{Coker}(f')$ *and* $\mathrm{Coker}(\lambda) \cong \mathrm{Coker}(\mu)$.

Proof. Consider the exact commutative diagram

$$\begin{array}{ccccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \vdots{\scriptstyle\nu} & & \\ L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0 \end{array}$$

Since $g'\mu f = g'f'\lambda = 0$, we see that there is a (unique) map ν with $\nu g = g'\mu$. On the other hand, since M' is a push-out, we obtain a (unique) map $\gamma\colon M' \to N$ such that $\gamma\mu = g$ and $\gamma f' = 0$. From this latter condition we obtain a (unique) map $\nu'\colon N' \to N$ such that $\gamma = \nu' g$.

We now check that ν and ν' are mutually inverse. We have

$$\nu'\nu g = \nu' g'\mu = \gamma\mu = g,$$

so since g is an epimorphism, $\nu'\nu = \mathrm{id}_N$. Similarly, $\nu\nu' g'\colon M' \to N'$ satisfies

$$\nu\nu' g'\mu = \nu\gamma\mu = \nu g = g'\mu \quad \text{and} \quad \nu\nu' g' f' = 0 = g' f',$$

so by the uniqueness part of the push-out property, $\nu\nu' g' = g'$. Then g' being an epimorphism implies $\nu\nu' = \mathrm{id}_{N'}$.

The proof that $\mathrm{Coker}(\lambda) \cong \mathrm{Coker}(\mu)$ is entirely analogous. □

The push-out along a composition is equivalent to the composition of the push-outs.

Lemma 4.6. *Given two push-out squares*

$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} \\ L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' \end{array}$$

the outer square is also a push-out

$$\begin{array}{ccc} L & \xrightarrow{gf} & N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\nu} \\ L' & \xrightarrow{g'f'} & N'. \end{array}$$

Proof. We check that the latter square satisfies the uniqueness property on maps.

Existence. Suppose we are given $\alpha\colon N \to X$ and $\beta\colon L' \to X$ such that $\alpha gf = \beta\lambda$. Then since M' is a push-out we have a unique map $\gamma\colon M' \to X$ with $\gamma f' = \beta$ and $\gamma\mu = \alpha g$, and then since N' is a push-out we have a unique map $\delta\colon N' \to X$ such that $\delta\nu = \alpha$ and $\delta g' = \gamma$, and hence also $\delta g' f' = \beta$.

Uniqueness. Here it is enough to show that $\delta\nu = 0$ and $\delta g' f' = 0$ implies $\delta = 0$. We have $\delta g'\mu = \delta\nu g = 0$, so $\delta g' = 0$ since M' is a push-out, and then $\delta = 0$ since N' is a push-out. □

Finally, push-outs of monomorphisms are again monomorphisms.

Lemma 4.7. *Suppose we have a short exact sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *and a map* $\lambda\colon L \to L'$*. Then we have an exact commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \| & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

if and only if the left hand square is a push-out.

Proof. Let E be the push-out. Since the cokernel of $L' \to E$ is isomorphic to the cokernel of f, namely N, there is an exact commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\mu} & & \| & & \\ & & L' & \xrightarrow{f'} & E & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

We need to show that f' is injective, so consider the exact sequence

$$L \to M \oplus L' \to E \to 0.$$

If $x \in L'$ is sent to zero, then $\binom{0}{x} \in M \oplus L'$ lies in $\mathrm{Ker}(\mu, f') = \mathrm{Im}\binom{-f}{\lambda}$, so $\binom{0}{x} = \binom{-f(l)}{\lambda(l)}$ for some $l \in L$. Since f is injective, we must have $l = 0$, whence $x = \lambda(l) = 0$.

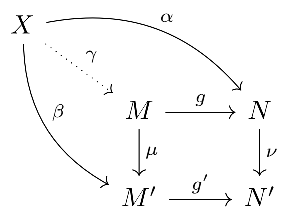
Now suppose we have such an exact commutative diagram with M'. Since E is a push-out we obtain a map $E \to M'$ fitting into an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L' & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

Applying the Snake Lemma to the bottom two rows we deduce that the map $E \to M'$ is an isomorphism. □

4.4 Pull-backs

We also have the dual notion. The pull-back of a pair of maps $\nu\colon N \to N'$ and $g'\colon M' \to N'$ is given by a commutative square satisfying the appropriate condition for maps from some module X



Equivalently, for all X there is an exact sequence

$$0 \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, N \oplus M') \to \mathrm{Hom}_R(X, N'),$$

given by composing with $\binom{g}{\mu}\colon M \to N \oplus M'$ and $(\nu, -g')\colon N \oplus M' \to N'$, or alternatively there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\binom{g}{\mu}} N \oplus M' \xrightarrow{(\nu,-g')} N'.$$

Thus a pull-back is the same as a kernel of the map $(\nu, -g')\colon N \oplus M' \to N'$, so they exist and are unique up to unique isomorphism.

In a pull-back square, parallel maps have naturally isomorphic kernels.

Lemma 4.8. *Given a pull-back square*

$$\begin{array}{ccc}
M & \xrightarrow{g} & N \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} \\
M' & \xrightarrow{g'} & N'
\end{array}$$

we have natural isomorphisms $\mathrm{Ker}(g) \cong \mathrm{Ker}(g')$ *and* $\mathrm{Ker}(\mu) \cong \mathrm{Ker}(\nu)$.

The pull-back along a composition is equivalent to the composition of the pull-backs.

Lemma 4.9. *Given two pull-back squares*

$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} \\ L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' \end{array}$$

the outer square is also a pull-back

$$\begin{array}{ccc} L & \xrightarrow{gf} & N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\nu} \\ L' & \xrightarrow{g'f'} & N'. \end{array}$$

Finally, pull-backs of epimorphisms are again epimorphisms.

Lemma 4.10. *Suppose we have a short exact sequence* $0 \to L' \xrightarrow{f'} M' \xrightarrow{g'} N' \to 0$ *and a map* $\nu\colon N \to N'$. *Then we have an exact commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

if and only if the right hand square is a pull-back.

4.5 The Krull-Remak-Schmidt Theorem

We begin with a nice lemma concerning pairs of short exact sequences with the same middle term.

Lemma 4.11. *Suppose we are given two short exact sequences with the same middle term*

$$0 \to X \xrightarrow{a} M \xrightarrow{b} X' \to 0 \quad \textit{and} \quad 0 \to Y \xrightarrow{c} M \xrightarrow{d} Y' \to 0.$$

Then these fit into an exact commutative diagram

$$\begin{array}{ccccc} E & \overset{i}{\rightarrowtail} & X & \twoheadrightarrow & \mathrm{Im}(da) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle a} & & \downarrow \\ Y & \overset{c}{\rightarrowtail} & M & \overset{d}{\twoheadrightarrow} & Y' \\ \downarrow & & \downarrow{\scriptstyle b} & & \downarrow \\ \mathrm{Im}(bc) & \rightarrowtail & X' & \twoheadrightarrow & F \end{array}$$

where E is the pull-back and F is the push-out.

Note. If $X, Y \leq M$ are submodules, then $E = X \cap Y$ and $F = M/(X+Y)$.

Proof. Let E be the pull-back of a and c, and write $i\colon E \to X$ and $j\colon E \to Y$ for the induced maps. We claim that $i\colon E \to X$ is a kernel for da.

Clearly $dai = dcj = 0$, so we need to show that, given $\alpha\colon W \to X$ with $da\alpha = 0$, there exists a unique $\gamma\colon W \to E$ with $\alpha = i\gamma$. The uniqueness follows since i is injective, by Lemma 4.8, so we just need to show existence.

Suppose $\alpha\colon W \to X$ satisfies $da\alpha = 0$. Since c is a kernel for d we can write $a\alpha = c\beta$. Then, using that E is a pull-back, we obtain $\gamma\colon W \to E$ with $i\gamma = a\alpha$ (and also $j\gamma = c\beta$).

Similarly j is a kernel for bc. Dually the push-out F yields cokernels for both da and bc. □

Corollary 4.12. *Suppose we have two short exact sequences*

$$0 \to X \xrightarrow{a} M \xrightarrow{b} X' \to 0 \quad \textit{and} \quad 0 \to Y \xrightarrow{c} M \xrightarrow{d} Y' \to 0.$$

Then da is an isomorphism if and only if bc is an isomorphism.

Proof. We have da is an isomorphism if and only if $E = 0 = F$, which is if and only if cb is an isomorphism. □

We can use this to prove the Krull-Remak-Schmidt Theorem.

Theorem 4.13 (Krull-Remak-Schmidt)**.** *Every finite length R-module can be written as a direct sum of indecomposable modules in an essentially unique way.*

Proof. Arguing by induction on length we see that we can always decompose a finite length module into a direct sum of indecomposable modules. We therefore just need to prove uniqueness.

Write $M \cong X_1 \oplus \cdots \oplus X_r$ with each X_i indecomposable. Associated to this decomposition we have the maps $X_i \xrightarrow{\iota_i} M \xrightarrow{\pi_i} X_i$. Now suppose we also have $M \cong Y \oplus Y'$ with Y indecomposable, and associated maps $Y \xrightarrow{f} M \xrightarrow{g} Y$, and similarly f', g' for Y'. Then

$$\mathrm{id}_Y = gf = g\big(\iota_1\pi_1 + \cdots + \iota_r\pi_r\big)f = \sum_i g\iota_i\pi_i f.$$

By Fitting's Lemma $\mathrm{End}_R(Y)$ is local, so without loss of generality we may assume $g\iota_r\pi_r f$ is an automorphism. We observe that $\pi_r f g\iota_r \in \mathrm{End}_R(X_r)$ cannot be nilpotent, so must be an automorphism by Fitting's Lemma again. Thus $\pi_r f$ is an automorphism, so we can apply the lemma to the short exact sequences

$$0 \to \mathrm{Ker}(\pi_r) \to M \xrightarrow{\pi_r} X_r \to 0 \quad \text{and} \quad 0 \to Y \xrightarrow{f} M \xrightarrow{g'} Y' \to 0$$

to obtain $Y' \cong \mathrm{Ker}(\pi_r) \cong X_1 \oplus \cdots \oplus X_{r-1}$. Now use induction on r. □

5 Extension groups

We write $\mathrm{Ext}^1_R(M,X)$ for the set of all equivalence classes of short exact sequences of the form

$$\varepsilon\colon 0 \to X \xrightarrow{f} E \xrightarrow{g} M \to 0.$$

Note that in a general abelian category, this may not actually be a set, but we will see shortly that this is the case in any module category.

Example 5.1. *Let $R = k[t]$. Then the module M is given by a pair (M,μ) consisting of a vector space M together with an endomorphism μ. Similarly a homomorphism $f\colon (M,\mu) \to (X,\xi)$ is given by a linear map $f\colon M \to X$ such that $f\mu = \xi f$.*

Now consider a short exact sequence

$$0 \to (X,\xi) \xrightarrow{f} (E,\tau) \xrightarrow{g} (M,\mu) \to 0.$$

Forgetting the action of t, this sequence must be split, so we have a vector space isomorphism $\alpha = \binom{r}{g}\colon E \xrightarrow{\sim} X \oplus M$, where r is any retract for f. Setting $\sigma := \alpha\tau\alpha^{-1}$, we have the equivalence of short exact sequences of $k[t]$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (X,\xi) & \xrightarrow{\quad f \quad} & (E,\tau) & \xrightarrow{\quad g \quad} & (M,\mu) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\
0 & \longrightarrow & (X,\xi) & \xrightarrow{\binom{1}{0}} & (X\oplus M,\sigma) & \xrightarrow{(01)} & (M,\mu) & \longrightarrow & 0.
\end{array}$$

Writing σ as a matrix we obtain $\sigma = \begin{pmatrix} \xi & \theta \\ 0 & \mu \end{pmatrix}$ for some linear map $\theta\colon M \to X$. We therefore write the class of this extension as $[\theta]$.

We now determine when $[\theta] = [\theta']$. Write $\sigma' = \begin{pmatrix} \xi & \theta' \\ 0 & \mu \end{pmatrix}$, and consider an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (X,\xi) & \xrightarrow{\binom{1}{0}} & (X\oplus M,\sigma) & \xrightarrow{(01)} & (M,\mu) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \longrightarrow & (X,\xi) & \xrightarrow{\binom{1}{0}} & (X\oplus M,\sigma') & \xrightarrow{(01)} & (M,\mu) & \longrightarrow & 0.
\end{array}$$

Then $\beta = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ for some linear map $f\colon M \to X$ such that $\beta\sigma = \sigma'\beta$, equivalently $\theta + f\mu = \xi f + \theta'$, or equivalently $\theta' = \theta + (f\mu - \xi f)$.

We conclude that there is a four term exact sequence

$$0 \to \mathrm{Hom}_{k[t]}(M,X) \to \mathrm{Hom}_k(M,X) \xrightarrow{\delta} \mathrm{Hom}_k(M,X) \xrightarrow{\pi} \mathrm{Ext}^1_{k[t]}(M,X) \to 0,$$

where $\delta(f) := \mu f - f\xi$ and $\pi(\theta) = [\theta]$.

5.1 The Baer sum

The push-out along $p\colon X \to Y$ yields a map

$$\mathrm{Ext}^1_R(M,X) \to \mathrm{Ext}^1_R(M,Y), \quad \varepsilon \mapsto p\varepsilon.$$

Similarly the pull-back along $f\colon L \to M$ yields a map

$$\mathrm{Ext}^1_R(M,X) \to \mathrm{Ext}^1_R(L,X), \quad \varepsilon \mapsto \varepsilon f.$$

The next lemma shows that these two operations are compatible.

Lemma 5.2. *We have* $p(\varepsilon f) = (p\varepsilon)f$ *for all* $f\colon L \to M$, $p\colon X \to Y$ *and* $\varepsilon \in \mathrm{Ext}^1_R(M, X)$.

Proof. We consider the commutative diagram with exact rows

$$\begin{array}{rccccccccc}
\varepsilon f\colon & 0 & \longrightarrow & X & \xrightarrow{\;r\;} & F & \xrightarrow{\;s\;} & L & \longrightarrow & 0\\
 & & & \| & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} & & \\
\varepsilon\colon & 0 & \longrightarrow & X & \xrightarrow{\;a\;} & E & \xrightarrow{\;b\;} & M & \longrightarrow & 0\\
 & & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} & & \| & & \\
p\varepsilon\colon & 0 & \longrightarrow & Y & \xrightarrow{\;t\;} & G & \xrightarrow{\;u\;} & M & \longrightarrow & 0.
\end{array}$$

Now take the pull-back along f of the bottom row, yielding a diagram with exact rows, and where the bottom half commutes

$$\begin{array}{rccccccccc}
\varepsilon f\colon & 0 & \longrightarrow & X & \xrightarrow{\;r\;} & F & \xrightarrow{\;s\;} & L & \longrightarrow & 0\\
 & & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q'} & & \| & & \\
(p\varepsilon)f\colon & 0 & \longrightarrow & Y & \xrightarrow{\;a'\;} & E' & \xrightarrow{\;b'\;} & L & \longrightarrow & 0\\
 & & & \| & & \big\downarrow{\scriptstyle g'} & & \big\downarrow{\scriptstyle f} & & \\
p\varepsilon\colon & 0 & \longrightarrow & Y & \xrightarrow{\;t\;} & G & \xrightarrow{\;u\;} & M & \longrightarrow & 0.
\end{array}$$

Using that E' is a pull-back, the map $q'\colon F \to E'$ is unique such that $b'q' = s$ and $g'q' = qg$, showing that the top right square commutes. We now compare the two maps $q'r$, $a'p\colon X \to E'$. We have

$$g'q'r = qgr = tp = g'a'p\colon X \to G \quad \text{and} \quad b'q'r = sr = 0 = b'a'p\colon X \to L.$$

Since also $utp = 0$, we can again use the uniqueness property for pull-backs to deduce that $q'r = a'p$, so the top left square commutes. It follows from Lemma 4.7 that the top left square is a push-out, so the extension class of the middle row also equals $p(\varepsilon f)$. □

We can now define an addition on $\mathrm{Ext}^1_R(M, X)$, called the Baer sum. Given two short exact sequences

$$\varepsilon\colon 0 \to X \xrightarrow{a} E \xrightarrow{b} M \to 0 \quad \text{and} \quad \varepsilon'\colon 0 \xrightarrow{a'} X \to E' \xrightarrow{b'} M \to 0,$$

it is easy to check that their direct sum is again a short exact sequence

$$\varepsilon \oplus \varepsilon'\colon \quad 0 \longrightarrow X^2 \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & a' \end{smallmatrix}\right)} E \oplus E' \xrightarrow{\left(\begin{smallmatrix} b & 0 \\ 0 & b' \end{smallmatrix}\right)} M^2 \longrightarrow 0.$$

We can then take the pull-back and push-out along the diagonal maps $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\colon M \to M^2$ and $(1\ 1)\colon X^2 \to X$ to obtain another element in $\mathrm{Ext}^1_R(M, X)$, denoted $\varepsilon + \varepsilon'$.

Proposition 5.3. *The Baer sum endows* $\mathrm{Ext}^1_R(M, X)$ *with the structure of an abelian group, with zero element the class of the split exact sequence.*

Proof. That the Baer sum is commutative is clear, since the direct sums $\varepsilon \oplus \varepsilon'$ and $\varepsilon' \oplus \varepsilon$ are equivalent.

For the associativity we note that to compute $\varepsilon + (\varepsilon' + \varepsilon'')$, we can start from the direct sum $\varepsilon \oplus \varepsilon' \oplus \varepsilon''$, take the push-out and pull-back along the maps

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} : X^3 \to X^2 \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} : M^2 \to M^3$$

to obtain $\varepsilon \oplus (\varepsilon' + \varepsilon'')$, and then take the push-out and pull-back along the maps

$$(1\ 1)\colon X^2 \to X \quad \text{and} \quad \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\colon M \to M^2.$$

Using that pull-backs and push-outs commute, and that the composition of push-outs is the same as the push-out of the composition, Lemma 4.6, and similarly for pull-backs, we see that this construction is the same as directly taking the push-out and pull-back along the maps

$$(1\ 1\ 1)\colon X^3 \to X \quad \text{and} \quad \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)\colon M \to M^3.$$

By symmetry this latter construction also computes $(\varepsilon + \varepsilon') + \varepsilon''$.

We check that the zero element is given by the class of the split exact sequence. This follows from the following commutative diagram with exact rows

$$\begin{array}{lccccccccc}
\varepsilon \oplus 0\colon & 0 & \longrightarrow & X^2 & \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} & E \oplus X \oplus M & \xrightarrow{\left(\begin{smallmatrix} b & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & M^2 & \longrightarrow & 0 \\
 & & & \| & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ b & 0 \end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} & & \\
 & 0 & \longrightarrow & X^2 & \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)} & E \oplus X & \xrightarrow{(b\ 0)} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle (1\ 1)} & & \downarrow{\scriptstyle (1\ a)} & & \| & & \\
\varepsilon + 0 = \varepsilon\colon & 0 & \longrightarrow & X & \xrightarrow{a} & E & \xrightarrow{b} & M & \longrightarrow & 0
\end{array}$$

Finally, the following two short exact sequences are additive inverses

$$\varepsilon\colon 0 \to X \xrightarrow{a} E \xrightarrow{b} M \to 0 \quad \text{and} \quad -\varepsilon\colon 0 \to X \xrightarrow{-a} E \xrightarrow{b} M \to 0.$$

For, consider the following commutative diagram with exact rows

$$\begin{array}{lccccccccc}
\varepsilon \oplus (-\varepsilon)\colon & 0 & \longrightarrow & X^2 & \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)} & E^2 & \xrightarrow{\left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)} & M^2 & \longrightarrow & 0 \\
 & & & \| & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 1 & -a \end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} & & \\
 & 0 & \longrightarrow & X^2 & \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 1 & 1 \end{smallmatrix}\right)} & E \oplus X & \xrightarrow{(b\ 0)} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle (1\ 1)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 1 \\ b & 0 \end{smallmatrix}\right)} & & \| & & \\
\varepsilon + (-\varepsilon)\colon & 0 & \longrightarrow & X & \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & X \oplus M & \xrightarrow{(0\ 1)} & M & \longrightarrow & 0
\end{array}$$

Note that in both diagrans the middle rows are exact, since they are both equivalent to the direct sum of ε and the trivial short exact sequence $0 \to X \xrightarrow{1} X \to 0 \to 0$. □

In fact, $\mathrm{Ext}^1_R(M,X)$ is naturally an $\mathrm{End}_R(X)$-$\mathrm{End}_R(M)$-bimodule via the push-out and pull-back constructions, and if R is a k-algebra, then the two k-module structures agree. We will not prove this directly, though, but deduce it from another construction. (The reason we are doing this is to save ourselves the trouble of discussing balanced functors.)

5.2 Projective modules

Given a k-algebra R, we can naturally regard R itself as a right R-module. This is called the regular module. A free module is one which is isomorphic to a direct sum of copies of the regular module, so $R^{(I)} := \bigoplus_I R$ for some set I.

Lemma 5.4. *We have* $\mathrm{Hom}_R(R^{(I)}, M) \cong M^I$*, sending* $f \mapsto (f(1_i))$*, where* $1_i \in R^{(I)}$ *has a* 1 *in position* i *and zeros elsewhere.*

Proof. We begin by observing that every element $x \in R^{(I)}$ is uniquely a finite R-linear combination of the 1_i. It follows that an R-module homomorphism $f\colon R^{(I)} \to M$ is uniquely determined by the elements $f(1_i) \in M$, and that these can be chosen arbitrarily. □

Lemma 5.5. *The following are equivalent for an* R*-module* P*.*

1. *Every short exact sequence* $0 \to L \to M \to P \to 0$ *splits.*
2. *If* $0 \to L \to M \to N \to 0$ *is exact, then so too is*

$$0 \to \mathrm{Hom}_R(P,L) \to \mathrm{Hom}_R(P,M) \to \mathrm{Hom}_R(P,N) \to 0.$$

3. P *is a direct summand of a free module.*

In this case we call P *a projective* R*-module.*

Note: Condition 2 can be rephrased as saying that $\mathrm{Hom}_R(P,-)$ is exact. Since $\mathrm{Hom}_R(P,-)$ is left exact, it is enough that we can lift maps from P along epimorphisms.

Proof. $2 \Rightarrow 1$. A lift of the identity map id_P gives a section to the epimorphism $M \twoheadrightarrow P$, so the sequence is split.

$1 \Rightarrow 3$. Every module is the quotient of a free module, and by assumption the epimorphism $R^{(I)} \twoheadrightarrow P$ is split.

$3 \Rightarrow 2$. We need to show $\mathrm{Hom}_R(P,M) \twoheadrightarrow \mathrm{Hom}_R(P,N)$. This is true for free modules, since given $g\colon M \twoheadrightarrow N$ and $h\colon R^{(I)} \to N$, say corresponding to $(n_i) \in N^I$, we just take any $m_i \in M$ mapping to n_i under g, and obtain the lift $h'\colon R^{(I)} \to M$ corresponding to $(m_i) \in M^I$.

In general, write $P \oplus Q \cong R^{(I)}$ and use the associated maps ι_P and π_P. Given $h\colon P \to N$ we have $h\pi_P\colon R^{(I)} \to N$, which we can lift to $h''\colon R^{(I)} \to M$. Then $h' := h''\iota_P$ is a lift of h. For, $gh' = gh''\iota_P = h\pi_P\iota_P = h$. □

Example 5.6. *Let* $e \in R$ *be an idempotent. Then* $R = eR \oplus e'R$*, so* eR *is a projective module. Moreover,* $\mathrm{Hom}_R(eR, M) \cong Me$ *via* $f \mapsto f(e)$*.*

5.3 Injective modules

The first two conditions for projective modules have their obvious duals. The third condition does not, but we can replace it by Baer's Criterion providing a simple test for injectivity.

Lemma 5.7. *The following are equivalent for an R-module I.*

1. *Every short exact sequence $0 \to I \to M \to N \to 0$ splits.*
2. *If $0 \to L \to M \to N \to 0$ is exact, then so too is*

$$0 \to \mathrm{Hom}_R(N, I) \to \mathrm{Hom}_R(M, I) \to \mathrm{Hom}_R(L, I) \to 0.$$

3. *(Baer's Criterion) It is enough to take $M = R$ in 2.*

In this case we call I an injective R-module.

Note: Condition 2 can be rephrased as saying that $\mathrm{Hom}_R(-, I)$ is exact. Since $\mathrm{Hom}_R(-, I)$ is left exact, it is enough that we can lift maps to I along monomorphisms.

Proof. $2 \Rightarrow 3$. Clear, since 3 is a special case of 2.

$1 \Rightarrow 2$. We need to show $\mathrm{Hom}_R(M, I) \twoheadrightarrow \mathrm{Hom}_R(L, I)$. Take a map $L \to I$ and consider the push-out

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

The bottom row is split, yielding a retract $r\colon E \to I$. It follows that the composition $M \to E \to I$ is a lift on the map $L \to I$.

$3 \Rightarrow 1$. Use Zorn's Lemma. We consider the set of pairs (U, r) such that $I \leq U \leq M$ and $r\colon U \to I$ is a retract of the inclusion. This is a poset, where we take $(U, r) \leq (U', r')$ provided $U \leq U'$ and $r'|_U = r$. It is non-empty since it contains (I, id_I). All chains have upper bounds, since if we have a chain (U_i, r_i), then we take $U = \bigcup_i U_i$ and define $r(u) := r_i(u)$ for any i such that $u \in U_i$. By Zorn's Lemma we have a maximal element (U, r).

Suppose $U < M$ and take $m \in M - U$. We have a map $f\colon R \to M$, $f(1) = m$. Take the right ideal $L < R$ consisting of those $a \in R$ such that $ma \in U$. Thus f restricts to a map $L \to U$, so we have $rf\colon L \to I$, which by assumption we can lift to $g\colon R \to I$.

It follows that there is a map $s\colon U + mR \to I$, $u + ma \mapsto r(u) + g(a)$. To show that s is well-defined consider $u + ma = v + mb$. Then $v - u = m(a - b) \in U$, so $a - b \in L$ and $g(a - b) = rf(a - b) = r(v - u)$. Thus $s(u + ma) = s(v + mb)$. Therefore $(U, r) < (U + mR, s)$, contradicting the maximality of (U, r). We conclude that $U = M$, so $I \rightarrowtail M$ is a split monomorphism. ☐

Example 5.8. *Let k be a field, R a k-algebra, and $D := \mathrm{Hom}_k(-, k)$ the usual vector space duality. If P is a projective left R-module, then $D(P)$ is an injective right R-module.*

5.4 Extension groups via projective resolutions

Lemma 5.9. *Let M and X be R-modules.*

1. *Let $\varepsilon\colon 0 \to U \xrightarrow{\iota} P \xrightarrow{\pi} M \to 0$ be exact, where P is projective. Then there is an exact sequence of additive groups*

   $$0 \to \mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(P,X) \to \mathrm{Hom}_R(U,X) \to \mathrm{Ext}^1_R(M,X) \to 0.$$

   This sends a map $p\colon U \to X$ to the push-out $p\varepsilon \in \mathrm{Ext}^1_R(M,X)$.

2. *Let $\eta\colon 0 \to X \to I \to V \to 0$ be exact, where I is injective. Then there is an exact sequence of additive groups*

   $$0 \to \mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(M,I) \to \mathrm{Hom}_R(M,V) \to \mathrm{Ext}^1_R(M,X) \to 0.$$

   This sends $f\colon M \to V$ to the pull-back $\eta f \in \mathrm{Ext}^1_R(M,X)$.

Proof. We prove the first, the second being dual.

We begin by showing that the push-out yields a map of additive groups. Given two maps $p,q\colon U \to X$, we first form their respective push-outs

$$\begin{array}{cccccc} \varepsilon\colon & U & \xrightarrow{\iota} & P & \xrightarrow{\pi} & M \\ & \downarrow p & & \downarrow p' & & \| \\ p\varepsilon\colon & X & \xrightarrow{a} & E & \xrightarrow{b} & M \end{array} \qquad \text{and} \qquad \begin{array}{cccccc} \varepsilon\colon & U & \xrightarrow{\iota} & P & \xrightarrow{\pi} & M \\ & \downarrow q & & \downarrow q' & & \| \\ q\varepsilon\colon & X & \xrightarrow{c} & F & \xrightarrow{d} & M \end{array}$$

Next, consider the following diagram, where the middle row is a push-out

$$\begin{array}{ccccc} U & \xrightarrow{\iota} & P & \xrightarrow{\pi} & M \\ \downarrow \binom{p}{q} & & \downarrow r & & \| \\ X^2 & \xrightarrow{f} & G & \xrightarrow{g} & M \\ \| & & \downarrow s & & \downarrow \binom{1}{1} \\ X^2 & \xrightarrow{\left(\begin{smallmatrix} a & \\ & c \end{smallmatrix}\right)} & E \oplus F & \xrightarrow{\left(\begin{smallmatrix} b & \\ & d \end{smallmatrix}\right)} & M^2 \end{array}$$

Using the push-out property there is a unique map $s\colon G \to E \oplus F$ such that $sf = \left(\begin{smallmatrix} a & \\ & c \end{smallmatrix}\right)$ and $sr = \binom{p'}{q'}$, and so the bottom left square commutes. We now compare the two maps $\binom{1}{1}g$, $\left(\begin{smallmatrix} b & 0 \\ 0 & d \end{smallmatrix}\right)s\colon G \to M^2$. We have

$$\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} sf = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = 0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} gf$$

and

$$\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} sr = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} gr.$$

Since also $\binom{1}{1}\pi\iota = 0$, we can again use the uniqueness property for pull-backs to deduce that $\left(\begin{smallmatrix} b & 0 \\ 0 & d \end{smallmatrix}\right)s = \binom{1}{1}g$, so the bottom right square commutes. It follows from Lemma 4.10 that the middle row is also a pull-back. If we now take the push-out of the middle row along the map $(1\ 1)\colon X^2 \to X$, we obtain $(p+q)\varepsilon = p\varepsilon + q\varepsilon$.

We next observe that the push-out of ε along any composition $p\iota$ with $p\colon P \to X$ is the split exact sequence

$$\begin{array}{ccccc} U & \overset{\iota}{\rightarrowtail} & P & \overset{\pi}{\twoheadrightarrow} & M \\ \downarrow{\scriptstyle p\iota} & & \downarrow{\scriptstyle \binom{p}{\pi}} & & \| \\ X & \overset{\binom{1}{0}}{\rightarrowtail} & X \oplus M & \overset{(0\ 1)}{\twoheadrightarrow} & M \end{array}$$

In particular, the push-out along the zero map is split, and so the push-out map is additive. To see that it is surjective, take any short exact sequence

$$0 \to X \xrightarrow{a} E \xrightarrow{b} M \to 0.$$

We can lift the map $\pi\colon P \to M$ along the epimorphism $E \twoheadrightarrow M$, and then using that a is a kernel we obtain a (unique) map $U \to X$ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

This is now a push-out diagram by Lemma 4.7.

Finally, the four term sequence is exact at $\mathrm{Hom}_R(U,X)$. For, we have already shown that the image of $\mathrm{Hom}_R(P,X)$ lies in the kernel. Conversely, given a push-out diagram

$$\begin{array}{ccc} U & \xrightarrow{\iota} & P \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & E \end{array}$$

if the bottom row is split, then we have a retract $r\colon E \to X$ for a, and hence $p = rap = rq\iota$ factors through ι. □

Corollary 5.10. *The additive group* $\mathrm{Ext}^1_R(M,X)$ *is naturally an* $\mathrm{End}_R(X)$*-*$\mathrm{End}_R(M)$*-bimodule. Moreover, if* R *is a* k*-algebra, then the two induced actions of* k *on* $\mathrm{Ext}^1_R(M,X)$ *agree.*

Proof. Consider the four term exact sequence

$$0 \to \mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(P,X) \to \mathrm{Hom}_R(U,X) \to \mathrm{Ext}^1_R(M,X) \to 0,$$

where $\varepsilon\colon 0 \to U \to P \to M \to 0$ is exact with P projective. Since the first three terms are naturally left $\mathrm{End}_R(X)$-modules, we obtain an induced $\mathrm{End}_R(X)$-module structure on $\mathrm{Ext}^1_R(M,X)$. In fact, this is just given by the usual push-out map. For, given any extension class, we can write it as a push-out $p\varepsilon$ for some $p\colon U \to X$. Then $f \in \mathrm{End}_R(X)$ acts on $\mathrm{Ext}^1_R(M,X)$ by $f \cdot p\varepsilon := (fp)\varepsilon$, which equals the composition of push-outs $f(p\varepsilon)$.

Similarly the right action of $\mathrm{End}_R(M)$ is given by the pull-back map, and since pull-backs and push-outs commute, we see that $\mathrm{End}_R(M,X)$ is naturally a bimodule.

Finally, suppose that R is a k-algebra. To see that the two actions of k agree, take $\lambda \in k$ and write λ_X for its image in $\mathrm{End}_R(X)$. Similarly for λ_E and λ_M. We then have the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a} & E & \xrightarrow{b} & M & \longrightarrow & 0 \\ && \downarrow{\scriptstyle\lambda_X} && \downarrow{\scriptstyle\lambda_E} && \downarrow{\scriptstyle\lambda_M} && \\ 0 & \longrightarrow & X & \xrightarrow{a} & E & \xrightarrow{b} & M & \longrightarrow & 0 \end{array}$$

As in the exercises, we have $\lambda_X\varepsilon = \varepsilon\lambda_M$, proving the claim.

Explicitly, consider the following commutative diagram, where the middle row is a pull-back

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a} & E & \xrightarrow{b} & M & \longrightarrow & 0 \\ && \downarrow{\scriptstyle\lambda_X} && \downarrow{\scriptstyle f} && \| && \\ 0 & \longrightarrow & X & \xrightarrow{c} & F & \xrightarrow{d} & M & \longrightarrow & 0 \\ && \| && \downarrow{\scriptstyle g} && \downarrow{\scriptstyle\lambda_M} && \\ 0 & \longrightarrow & X & \xrightarrow{a} & E & \xrightarrow{b} & M & \longrightarrow & 0 \end{array}$$

Since the two actions of k on $\mathrm{Hom}_R(E, M)$ agree, we know that $\lambda_M b = b\lambda_E$. Then, since F is a pull-back, we have a unique map $f\colon E \to F$ such that $df = b$ and $gf = \lambda_E$. We then check

$$dfa = ba = 0 = dc\lambda_X \quad \text{and} \quad gfa = \lambda_E a = a\lambda_X = gc\lambda_X,$$

so by uniqueness we have $fa = c\lambda_X$. Therefore the top half of the diagram commutes, so the middle row is also a push-out by Lemma 4.7. $\square$

5.5 Long exact sequence for ext

Lemma 5.11. *Let $\varepsilon\colon 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be exact. Then for all X there is a long exact sequence (of left $\mathrm{End}_R(X)$-modules)*

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_R(N, X) \xrightarrow{g^*} \mathrm{Hom}_R(M, X) \xrightarrow{f^*} \mathrm{Hom}_R(L, X) \xrightarrow{\varepsilon^*} \\ \qquad \mathrm{Ext}^1_R(N, X) \xrightarrow{g^*} \mathrm{Ext}^1_R(M, X) \xrightarrow{f^*} \mathrm{Ext}^1_R(L, X) \end{array}$$

Here the map ε^* sends $\lambda\colon L \to X$ to the push-out $\lambda\varepsilon$, and g^* acts on extensions as the pull-back map $\varepsilon' \mapsto \varepsilon' g$.

Proof. This follows from the Snake Lemma. More precisely, take epimorphisms $p_L\colon P_L \twoheadrightarrow L$ and $p_N\colon P_N \twoheadrightarrow N$ with P_L and P_N projective. Since P_N is projective we can lift the map $P_N \to N$ to a map $q\colon P_N \to M$. Now set $P_M := P_L \oplus P_N$, and $p_M := fp_L\pi_L + q\pi_N\colon P_M \to M$. Then P_M is projective, and the Snake Lemma yields an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_L & \xrightarrow{a} & U_M & \xrightarrow{b} & U_N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_L} & & \downarrow{\scriptstyle i_M} & & \downarrow{\scriptstyle i_N} & & \\
0 & \longrightarrow & P_L & \xrightarrow{\iota_L} & P_M & \xrightarrow{\pi_N} & P_N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_L} & & \downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle p_N} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We now apply $\mathrm{Hom}_R(-, X)$ to the top two rows, and use that the middle row is split exact, to get an exact commutative diagram (of $\mathrm{End}_R(X)$-modules)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(P_N, X) & \xrightarrow{\pi_N^*} & \mathrm{Hom}_R(P_M, X) & \xrightarrow{\iota_L^*} & \mathrm{Hom}_R(P_L, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_N^*} & & \downarrow{\scriptstyle i_M^*} & & \downarrow{\scriptstyle i_L^*} & & \\
0 & \longrightarrow & \mathrm{Hom}_R(U_N, X) & \xrightarrow{b^*} & \mathrm{Hom}_R(U_M, X) & \xrightarrow{a^*} & \mathrm{Hom}_R(U_L, X). & &
\end{array}$$

We now apply the Snake Lemma to this diagram to obtain a long exact sequence

$$\begin{array}{l}
0 \longrightarrow \mathrm{Hom}_R(N, X) \xrightarrow{\alpha} \mathrm{Hom}_R(M, X) \xrightarrow{\beta} \mathrm{Hom}_R(L, X) \\
\xrightarrow{\delta} \mathrm{Ext}^1_R(N, X) \xrightarrow{\theta} \mathrm{Ext}^1_R(M, X) \xrightarrow{\phi} \mathrm{Ext}^1_R(L, X)
\end{array}$$

It remains to compute the morphisms. Suppose $\nu\colon N \to X$. Then $\alpha(\nu)\colon M \to X$ is uniquely determined by $\alpha(\nu)p_M = \nu p_N \pi_N$. Since $p_N\pi_N = gp_M$, we get $\alpha(\nu) = \nu g$, so $\alpha = g^*$. Similarly $\beta = f^*$.

Next, given $\nu\colon U_N \to X$, we have $\theta(\nu\varepsilon_N) := b^*(\nu)\varepsilon_M = \nu b\varepsilon_M$. As in exercises, we know $b\varepsilon_M = \varepsilon_N g$. Thus $\theta(\nu\varepsilon_N) = \nu\varepsilon_N g$, and so θ is just pull-back along g. Similarly ϕ is just the pull-back along f.

Explicitly: We now construct the following commutative diagram with exact rows

$$\begin{array}{cccccccccc}
\varepsilon_M\colon & 0 & \longrightarrow & U_M & \xrightarrow{i_M} & P_M & \xrightarrow{p_M} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle b} & & \downarrow & & \| & & \\
 & 0 & \longrightarrow & U_N & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\
\varepsilon_N\colon & 0 & \longrightarrow & U_N & \xrightarrow{i_N} & P_N & \xrightarrow{p_N} & N & \longrightarrow & 0
\end{array}$$

Thus the middle row is both the push-out and the pull-back, so $b\varepsilon_M = \varepsilon_N g$.

Finally, we check that the connecting homomorphism is given by the push-out. Consider the exact commutative diagram

$$\begin{array}{rccccccccc} \varepsilon_N\colon & 0 & \longrightarrow & U_N & \xrightarrow{i_N} & P_N & \xrightarrow{p_N} & N & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} & & \| & & \\ \varepsilon\colon & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

where r is unique such that $fr = qi_N$, using that f is a kernel. In particular, $\varepsilon = r\varepsilon_N$.

Now take any $\lambda\colon L \to X$. Recall that the connecting map in the Snake Lemma sends λ to $\nu\varepsilon_N$, whenever $\nu\colon U_N \to X$ and $\mu\colon P_M \to X$ satisfy both $\mu i_M = \nu b$ and $\mu\iota_L = \lambda p_L$. We claim that we can take $\mu := \lambda p_L \pi_L$ and $\nu = -\lambda r$. For, it is clear that $\mu\iota_L = \lambda p_L \pi_L \iota_L = \lambda p_L$. Also, using that $p_M = f p_L \pi_L + q\pi_N$, we have

$$frb = q\pi_N i_M = (p_M - f p_L \pi_L) i_M = -f p_L \pi_L i_M,$$

and since f is injective we obtain $rb = -p_L \pi_L i_M$. Thus $\mu i_M = \lambda p_L \pi_L i_M = -\lambda rb = \nu b$. So the connecting homomorphism from the Snake Lemma sends λ to the push-out $-\lambda r\varepsilon_N = -\lambda\varepsilon$. Since the minus sign does not change exactness, we can take the connecting map $\lambda \mapsto \lambda\varepsilon$ as claimed. □

Remark. If a^* is onto, for example if U_N is projective, then so too is the map f^* on extension classes.

There is also the dual result.

Lemma 5.12. *Let $\varepsilon\colon 0 \to W \xrightarrow{f} X \xrightarrow{g} Y \to 0$ be exact. Then for all M there is a long exact sequence (of $\mathrm{End}_R(M)$-modules)*

$$0 \longrightarrow \mathrm{Hom}_R(M,W) \xrightarrow{f_*} \mathrm{Hom}_R(M,X) \xrightarrow{g_*} \mathrm{Hom}_R(M,Y) \xrightarrow{\varepsilon_*} \mathrm{Ext}^1_R(M,W) \xrightarrow{f_*} \mathrm{Ext}^1_R(M,X) \xrightarrow{g_*} \mathrm{Ext}^1_R(M,Y)$$

Here the map ε_* sends $\eta\colon M \to Y$ to the pull-back $\varepsilon\eta$, and f_* acts on extensions as the push-out map $\varepsilon' \mapsto f\varepsilon'$.

6 Hereditary algebras

We can now introduce the notion of an hereditary algebra.

Proposition 6.1. *The following are equivalent for a k-algebra R.*

1. *All right ideals of R are projective.*
2. *Submodules of projective modules are themselves projective.*
3. *If $L \rightarrowtail M$, then $\mathrm{Ext}^1_R(M,X) \twoheadrightarrow \mathrm{Ext}^1_R(L,X)$ for all X.*
4. *Quotients of injective modules are themselves injective.*
5. *If $X \twoheadrightarrow Y$, then $\mathrm{Ext}^1_R(M,X) \twoheadrightarrow \mathrm{Ext}^1_R(M,Y)$ for all M.*
6. *For all $f\colon M \to N$ there exists a module E fitting into a short exact sequence*
$$0 \to M \to E \oplus \mathrm{Im}(f) \to N \to 0.$$

In this case we call R hereditary (since being projective is inherited by submodules).

Proof. $2 \Rightarrow 3$. Take two short exact sequences $0 \to L \xrightarrow{f} M \to N \to 0$ and $0 \to U_N \to P_N \to N \to 0$, with P_N projective. Then U_N is projective, so by Lemma 5.11 and the subsequent remark we see that $f^*\colon \mathrm{Ext}^1_R(M,X) \to \mathrm{Ext}^1_R(L,X)$ is onto for all X.

$3 \Rightarrow 6$. We can factor f via its image, so $M \overset{\pi}{\twoheadrightarrow} \mathrm{Im}(f) \overset{\iota}{\rightarrowtail} N$ with $f = \iota\pi$. By assumption the pull-back map

$$\iota^*\colon \mathrm{Ext}^1_R(N, \mathrm{Ker}(f)) \to \mathrm{Ext}^1_R(\mathrm{Im}(f), \mathrm{Ker}(f))$$

is onto, so we have an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & M & \xrightarrow{\pi} & \mathrm{Im}(f) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & E & \xrightarrow{b} & N & \longrightarrow & 0
\end{array}$$

Since M is a pull-back, and $b\colon E \to N$ is onto, we have the short exact sequence

$$0 \longrightarrow M \xrightarrow{\binom{-a}{\pi}} E \oplus \mathrm{Im}(f) \xrightarrow{(b,\iota)} N \longrightarrow 0.$$

$6 \Rightarrow 2$. Let N be projective, $U \leq N$ a submodule, and write $M \twoheadrightarrow U$ with M free (or just projective). Then U is the image of $M \to N$, and hence there exists E with $0 \to M \to E \oplus U \to N \to 0$ exact. This is split, since N is projective, so U is a summand of $M \oplus N$, and hence is itself projective by Lemma 5.5.

Dually 4, 5 and 6 are equivalent.

$2 \Rightarrow 1$. Clear.

$1 \Rightarrow 4$. Let I be injective, and $\theta\colon I \twoheadrightarrow J$ an epimorphism. We will use Baer's Criterion from Lemma 5.7 to prove that J is injective. To this end, let $L \leq R$ be a right ideal, and $f\colon L \to J$ any homomorphism. Since L is projective we can lift f to a map $g\colon L \to I$, so $f = \theta g$. Next, since I is injective, we can lift g to $h\colon R \to I$, so $g = hi$, where $i\colon L \to R$ is the inclusion map. Then $\theta h\colon R \to J$ satisfies $\theta h i = \theta g = f$, and we are done. □

6.1 Commutative hereditary rings

Lemma 6.2. *Let R be any hereditary ring. Then its centre Z is reduced (has no non-zero nilpotent elements).*

Proof. Take $z \in Z$ and consider multiplication by z as a map $R \to R$. The image zR is projective, so we can write $R = K \oplus I$ where $K = \{a \in R : za = 0\}$ is the kernel and $I \xrightarrow{\sim} zR$. Note that I is a right ideal, and K is a two-sided ideal. Write $1 = a + x$ with $a \in K$ and $x \in I$. Then $z = az + xz = xz$ lies in I. If now z is nilpotent, say $z^{n+1} = 0$, then also $z^n \in K \cap I = 0$. We conclude that $z = 0$, so Z is reduced. □

In particular, every commutative hereditary ring is necessarily reduced.

We next observe that every principal ideal domain is hereditary. For, if $I \lhd R$ is a non-zero ideal, then $I = aR$ for some a, and since R is a domain, multiplication by a yields an isomorphism $R \xrightarrow{\sim} aR$. Thus $I = aR$ is a free module.

More generally, we have the following result.

Theorem 6.3. *Let R be a commutative domain. Then R is hereditary if and only if it is a Dedekind domain.*

Proof. Proof omitted. □

6.2 Tensor algebras

Let A be any k-algebra, and M an A-bimodule on which k acts centrally. We can then form the tensor algebra

$$R = T_A(M) := \bigoplus_{n\geq 0} M^{\otimes_A n} = A \;\oplus M \;(M \otimes_A M)\;(M \otimes_A M \otimes_A M)\;\cdots.$$

The multiplication is given by concatenation of tensors, so is induced by the usual isomorphism

$$M^{\otimes m} \otimes_A M^{\otimes n} \xrightarrow{\sim} M^{\otimes(m+n)}.$$

Note that A is a subalgebra. Also $R_+ := \bigoplus_{n\geq 1} M^{\otimes n}$ is a two-sided ideal, and $R/R_+ \cong A$.

Example 6.4. *Let $A = k$ be a field, and $M = k$ a one-dimensional vector space. Then the tensor algebra is just the polynomial algebra in one variable, $k[t]$. For, we let t be any basis vector for M. Then $M^{\otimes n} \cong k$ with basis vector t^n.*

More generally, if $\dim M = m$*, then the tensor algebra is isomorphic to the free algebra on m variables $k\langle t_1, \dots, t_m\rangle$. For, if $t_1, \dots, t_m$ is a basis for M, then $M \otimes_k M$ has basis $t_i t_j$ for all pairs i, j, and in general $M^{\otimes n}$ has basis given by all words of length n in the t_i (so has dimension m^n).*

Lemma 6.5. *A module over the tensor algebra $T_A(M)$ is the same as a pair (X, ξ), where X is an A-module and $\xi\colon X \otimes_A M \to X$ is an A-module homomorphism. In this setting, a $T_A(M)$-module homomorphism $f\colon (X, \xi) \to (Y, \eta)$ is the same as a an A-module homomorphism $f\colon X \to Y$ such that $f\xi = \eta(f \otimes 1)$.*

Proof. Given a $T_A(M)$-module X, its restriction to A is an A-module, and the action of M induces an A-module homomorphism $\xi\colon X \otimes_A M \to X$. Conversely, given a pair (X, ξ), we recursively obtain maps $\xi_n\colon X \otimes_A M^{\otimes n} \to X$, so $\xi_n = \xi(\xi_{n-1} \otimes \mathrm{id}_M)$. These then combine to yield a map $X \otimes_A T_A(M) \to X$, which endows X with the structure of a $T_A(M)$-module. Moreover, these constructions are mutually inverse.

Now let $f\colon X \to Y$ be a $T_A(M)$-module homomorphism. Again, restriction to A shows that f is an A-module homomorphism, and considering the action of M we have $f(x \cdot m) = f(x) \cdot m$ for all $x \in X$ and $m \in M$, equivalently $f\xi = \eta(f \otimes \mathrm{id}_M)$. Conversely, suppose we have a morphism $f\colon (X, \xi) \to (Y, \eta)$. Then the $T_A(M)$-module structure on X is given by the maps ξ_n as above, and we see that $f\xi_n = \eta_n(f \otimes \mathrm{id}_{M^{\otimes n}})$. Thus f is a homomorphism of $T_A(M)$-modules. Moreover, these constructions are again mutually inverse. □

Example 6.6. *Let $A = k$ and $M = k$, so that $T_A(M) = k[t]$. Then a $k[t]$-module is the same as a k-module X together with a map $X \cong X \otimes_k k \to X$, so an endomorphism $T \in \mathrm{End}_k(X)$, giving the action of t. A homomorphism $(X, T) \to (Y, U)$ is a k-module homomorphism $f\colon X \to Y$ which intertwines the endomorphisms, so $fT = Uf$.*

More generally, if $\dim M = m$, then $T_A(M) = k\langle t_1, \dots, t_m\rangle$ is the free algebra on m variables, and a module for the free algebra is the same as a k-module X together with an m-tuple of endomorphisms $(T_1, \dots, T_m)$ of X. A homomorphism $(X, T_1, \dots, T_m) \to (Y, U_1, \dots, U_m)$ is a k-module homomorphisms $f\colon X \to Y$ such that $fT_i = U_i f$ for all i.

6.3 The standard exact sequence

Every module over a tensor algebra fits into a standard short exact sequence.

Proposition 6.7. *Let $T_A(M)$ be a tensor algebra, and X a $T_A(M)$-module. Then there is a short exact sequence*

$$0 \to X \otimes_A M \otimes_A T_A(M) \xrightarrow{\delta} X \otimes_A T_A(M) \xrightarrow{\mu} X \to 0,$$

where μ is just the module structure and $\delta = \mathrm{id}_X \otimes i - \mu \otimes \mathrm{id}_{T_A(M)}$, where $i\colon M \otimes_A T_A(M) \rightarrowtail T_A(M)$ is the obvious inclusion.

Proof. We consider the split exact sequence of left A-modules

$$0 \longrightarrow X \otimes_A M \otimes_A T_A(M) \xrightarrow{\mathrm{id}_X \otimes i} X \otimes_A T_A(M) \xrightarrow{\ \pi\ } X \longrightarrow 0,$$

where π sends $x \otimes (a + m + \cdots)$ to xa.

Next, we have the locally nilpotent endomorphism θ of $X \otimes_A T_A(M)$, sending $x \otimes (m_1 \otimes \cdots \otimes m_n) \in X \otimes_A M^{\otimes n}$ to $xm_1 \otimes (m_2 \otimes \cdots \otimes m_n) \in X \otimes_A M^{\otimes(n-1)}$. Then $1 - \theta$ is an automorphism of $X \otimes_A T_A(M)$, and we have both $\mu(1-\theta) = \pi$ and $(1-\theta)(\mathrm{id}_X \otimes i) = \delta$. □

An algebra A is semisimple provided the regular module is a direct sum of simple modules, which are then necessarily projective. Schur's Lemma then implies that every simple module is projective. It follows that every finitely generated module is semisimple, and in fact by Zorn's Lemma, every module is semisimple.

Theorem 6.8 (Artin-Wedderburn). *An algebra is semisimple if and only if it is isomorphic to a (finite) direct product of matrices over division rings.*

Proof. Omitted. □

Proposition 6.9. *Let A be semisimple, and M an A-bimodule. Then the tensor algebra $R = T_A(M)$ is hereditary.*

Proof. Let $I \le R$ be a right ideal, set $X = R/I$, and consider the following exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X \otimes_A M \otimes R & \longrightarrow & X \otimes_A R & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

where the bottom row is the standard exact sequence and the map $R \to X \otimes_A R$ exists since R is projective. Then the left hand square is a push-out, and since $I \le R$, we have a short exact sequence

$$0 \to I \to R \oplus (X \otimes_A M \otimes_A R) \to X \otimes_A R \to 0.$$

Now, X_A is a projective right A-module, so $X \otimes_A T_A(M)$ is a projective right $T_A(M)$-module. (For, if $X \oplus Y \cong A^{(I)}$, then $(X \otimes_A T_A(M)) \oplus (Y \otimes_A T_A(M)) \cong T_A(M)^{(I)}$.) Similarly $X \otimes_A M \otimes_A T_A(M)$ is a projective $T_A(M)$-module. Thus the above short exact sequence must split, so I is a direct summand of the middle term, which is projective. Hence I is itself projective, and so R is hereditary. □

We can use the standard exact sequence to obtain a four term exact sequence relating Hom and Ext. Note that this generalises the four term sequence described in Example 5.1.

Corollary 6.10. *Let A be semisimple, and $R = T_A(M)$ a tensor algebra. Given R-modules X and Y we have the four term exact sequence*

$$0 \to \mathrm{Hom}_R(X,Y) \to \mathrm{Hom}_A(X,Y) \xrightarrow{\partial} \mathrm{Hom}_A(X \otimes_A M, Y) \to \mathrm{Ext}^1_R(X,Y) \to 0.$$

Here, if $f\colon X \to Y$ is an A-module homomorphism, then $\partial(f)(x \otimes m) := f(x)m - f(xm)$.

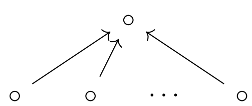
Proof. Apply $\mathrm{Hom}_R(-,Y)$ to the standard exact sequence for X. Then use that

$$\mathrm{Hom}_R(X \otimes_A R, Y) \cong \mathrm{Hom}_A(X, \mathrm{Hom}_R(R,Y)) \cong \mathrm{Hom}_A(X,Y),$$

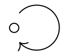
and similarly for $\mathrm{Hom}_R(X \otimes_A M \otimes_A R, Y)$. □

6.4 Path algebras of quivers

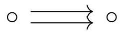
A quiver Q consists of a finite set of vertices Q_0 and a finite set of arrows Q_1, where each arrow $a\colon s(a) \to t(a)$ starts at $s(a)$ and has tip at $t(a)$. Examples include the n-subspace quiver



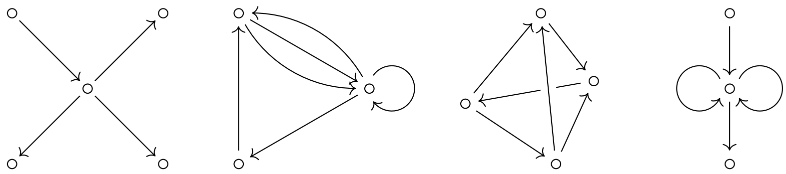
the Jordan quiver



and the Kronecker quiver



but we could also have more complicated quivers such as



Let Q be a finite quiver. We first set $A := k^{Q_0}$, and note that this is a semisimple k-algebra, with a complete set of orthogonal idempotents ε_i indexed by the vertices $i \in Q_0$. We then set $M := kQ_1$, a vector space having basis the arrows of Q. We give M the structure of an A-bimodule by setting $\varepsilon_i a \varepsilon_j = a$ provided $i = s(a)$ and $j = t(a)$, and is zero otherwise, and then extending linearly so that k acts centrally on M.

Consider $M \otimes_A M$. Decomposing $1 = \sum_i \varepsilon_i$ in A, we have

$$M \otimes_A M \cong \bigoplus_{h,i,j} \varepsilon_h M \varepsilon_i \otimes_k \varepsilon_i M \varepsilon_j.$$

Thus, as a vector space, this has basis ab such that $a, b \in Q_1$ are arrows, and $t(a) = s(b)$.

In general, a path of length $r \geq 1$ in Q is a sequence $p = a_1 \cdots a_r$ of arrows $a_i \in Q_1$ such that $t(a_i) = s(a_{i+1})$ for all $1 \leq i < r$. We call the idempotents ε_i the paths of length zero. Then $M^{\otimes r}$ has basis the paths of length r, for all $r \geq 0$, and so the path algebra $kQ := T_A(M)$ is an hereditary k-algebra with basis the set of all paths in Q. In particular, kQ is finite dimensional if and only if there are no oriented cycles in Q.

We remark that this definition of paths is the opposite of that given by (most) other authors, the reason being that we are dealing with right modules but want the simple projective modules to correspond to the sinks in the quiver.

Example 6.11. *Let Q be the Jordan quiver. Then $kQ \cong k[t]$, where t corresponds to the loop.*

Let Q be the n-subspace quiver. Then kQ is isomorphic to the subalgebra of $\mathbb{M}_{n+1}(k)$ given by matrices of the form

$$\begin{pmatrix} k & & & & 0 \\ k & k & & & \\ k & 0 & k & & \\ \vdots & & \ddots & \ddots & \\ k & 0 \cdots & & 0 & k \end{pmatrix}$$

Let Q be the Kronecker quiver. Then kQ is given by the generalised matrix algebra

$$\begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$$

Consider now modules over path algebras. As for any tensor algebra, the modules correspond to pairs (X,ξ) where X is an A-module and $\xi\colon X\otimes_A M\to X$ is an A-module homomorphism. For the path algebra, we have $A=k^{Q_0}$, so A-modules correspond to tuples of k-vector spaces, indexed by the vertices, so we can replace X by vector spaces X_i for each vertex i. Next, we know that M has basis given by the arrows, and so ξ is completely determined by the linear maps $\xi_a\colon X_i\to X_j$ for each arrow $a\colon i\to j$. Thus kQ-modules correspond to the data (X_i,ξ_a), often referred to as a k-representation of the quiver Q. In this language, a homomorphism $f\colon (X_i,\xi_a)\to(Y_i,\eta_a)$ is given by a tuple of linear maps $f_i\colon X_i\to Y_i$ such that $f_j\xi_a=\eta_a f_i$ for all $a\colon i\to j$; in other words, for each arrow $a\colon i\to j$ we have a commutative square

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow{\scriptstyle \xi_a} & & \downarrow{\scriptstyle \eta_a} \\ X_j & \xrightarrow{f_j} & Y_j \end{array}$$

Finally, let (X_i,ξ_a) and (Y_i,η_a) be two quiver representations. Then the map δ from Corollary 6.10 becomes

$$\bigoplus_i \mathrm{Hom}_k(X_i,Y_i) \xrightarrow{\partial} \bigoplus_{a\colon i\to j} \mathrm{Hom}_k(X_i,Y_j), \qquad \partial(f_i)=(\eta_a f_i - f_j\xi_a).$$

6.5 Simple modules and projective modules

Let R be any ring, and $\varepsilon\in R$ an idempotent. Recall that εR is a projective module, and that $\mathrm{Hom}_R(\varepsilon R,M)\cong M\varepsilon$, via the map $f\mapsto f(\varepsilon)$. We say that ε is primitive if εR is indecomposable.

Lemma 6.12. *An idempotent ε is primitive if and only if it is the only non-zero idempotent in $\varepsilon R\varepsilon$.*

Proof. We have seen that direct summands of a module M correspond to idempotents in its endomorphism ring $\mathrm{End}_R(M)$. One direction sends a summand L to the idempotent $\iota_L\pi_L$, whereas if e is an idempotent, then $M=\mathrm{Im}(e)\oplus\mathrm{Ker}(e)$. The result follows, using that $\mathrm{End}_R(\varepsilon R)\cong\varepsilon R\varepsilon$. □

Now let $R=T_A(M)$ be an hereditary tensor algebra. Note that $R_+=\bigoplus_{n\geq 1}M^{\otimes n}$, so $R_+^t=\bigoplus_{n\geq t}M^{\otimes n}$, and hence $\bigcap_{t\geq 1}R_+^t=0$. We assume moreover that A is a basic, semisimple k-algebra, so $A=\prod_i A_i$ is a finite product of division algebras.[1] Let $1=\sum_i\varepsilon_i$ be the corresponding decomposition into (primitive, orthogonal, central) idempotents in A. Then $A_i=\varepsilon_i A$ is a simple A-module, which we may regard as an R-module S_i via the epimorphism $R\twoheadrightarrow A$. Similarly, since $A\leq R$ is a subalgebra, we have $1=\sum_i\varepsilon_i$ as a sum of (orthogonal) idempotents in R, and hence $P_i:=\varepsilon_i R$ is a projective R-module.

Theorem 6.13. *Let A be a basic semisimple algebra, and $R=T_A(M)$ an hereditary tensor algebra. Write $1=\sum_i\varepsilon_i$ as a sum of primitive idempotents in A. Then*

[1] This is not a serious restriction, as every algebra is Morita equivalent to a basic algebra, and an hereditary tensor algebra is basic if and only if A is basic.

1. *The P_i are indecomposable projective R-modules, and $S_i \cong P_i/P_iR_+$.*

2. *The S_i are simple R-modules, and* $\mathrm{End}_R(S_i) \cong A_i$. *Also,*

$$\mathrm{Ext}^1_R(S_i, S_j) \cong \varepsilon_j M^\vee \varepsilon_i, \quad \textit{where } M^\vee := \mathrm{Hom}_A(M_A, A_A).$$

3. *If R_+ is nilpotent, then* $\mathrm{End}_R(P_i) \cong A_i$.

Proof. (1) By the lemma, to see that P_i is indecomposable, we need to show that there is no non-zero idempotent in $\varepsilon_i R \varepsilon_i$ other than ε_i itself. Suppose $x \in \varepsilon_i R \varepsilon_i$ is idempotent; then so too is $y := \varepsilon_i - x$. Write $x = x_0 + x_1 \in A \oplus R_+$, and similarly $y = y_0 + y_1$. Then $x_0, y_0 \in \varepsilon_i A \varepsilon_i = A_i$ are idempotents, and $x_0 + y_0 = \varepsilon_i$. Since A_i is a division ring, the only idempotents are ε_i and 0, so without loss of generality we may assume $y_0 = 0$. Then $y = y_1 \in R_+$ is an idempotent, so $y = y^n \in R_+^n$, hence $y \in \bigcap_{n\geq 1} R_+^n = 0$. Thus $x = \varepsilon_i$.

Now consider the standard exact sequence for $S_i = \varepsilon_i A$

$$0 \to \varepsilon_i A \otimes_A R_+ \to \varepsilon_i A \otimes_A R \to S_i \to 0,$$

and observe that $\varepsilon_i A \otimes_A R = \varepsilon_i = P_i$, and similarly $\varepsilon_i A \otimes_A R_+ = \varepsilon_i R_+ = P_i R_+$.

(2) Since A_i is a division algebra, it is clearly simple as an A-module, and since $R \twoheadrightarrow A$, every R-submodule is necessarily an A-submodule. Thus S_i is a simple R-module.

Now, since $S_i R_+ = 0$, we see that for any R-module X, the submodule XR_+ is contained in the kernel of every homomorphism $X \to S_j$. It follows that

$$\mathrm{Hom}_R(X, S_j) \cong \mathrm{Hom}_R(X/XR_+, S_j) \cong \mathrm{Hom}_A(X/XR_+, S_j).$$

In particular, we have

$$\mathrm{Hom}_R(P_i, S_j) \cong \mathrm{Hom}_A(S_i, S_j) \cong \varepsilon_j A \varepsilon_i,$$

and this is zero unless $i = j$ in which case we get A_i.

Applying $\mathrm{Hom}_R(-, S_j)$ to the standard exact sequence now yields

$$\mathrm{Ext}^1_R(S_i, S_j) \cong \mathrm{Hom}_R(P_i R_+, S_j) \cong \mathrm{Hom}_A(\varepsilon_i M, S_j),$$

where we have used that

$$P_i R_+ / P_i R_+^2 = \varepsilon_i R_+ / \varepsilon_i R_+^2 \cong \varepsilon_i (R_+/R_+^2) \cong \varepsilon_i M.$$

Finally we observe that $M^* = \mathrm{Hom}_A(M_A, A_A)$ is naturally an A-bimodule, via $(afa')(m) = a \cdot f(a'm)$. This yields the map $\varepsilon_j M^* \varepsilon_i \to \mathrm{Hom}_A(\varepsilon_i M, A_j)$, with inverse sending g to $\hat{g}(m) := g(\varepsilon_i m)$.

(3) We have $\mathrm{End}_R(P_i) \cong \varepsilon_i R \varepsilon_i = A_i \oplus \varepsilon_i R_+ \varepsilon_i$, and every element in $\varepsilon_i R_+ \varepsilon_i$ corresponds to a nilpotent endomorphism. On the other hand, every non-zero endomorphism of P_i is necessarily a monomorphism, by Lemma 6.16. Thus $\varepsilon_i R_+ \varepsilon_i = 0$, and hence $\mathrm{End}_R(P_i) \cong A_i$. □

Example 6.14. *Let Q be a quiver. Then the simple S_i corresponds to the representation having vector space k at vertex i and zeros elsewhere, and all linear maps zero. The projective P_i corresponds to the representation whose vector space at vertex j has basis all paths from i to j, and where an arrow*

$a\colon j \to l$ sends a basis vector p, a path from i to j, to the basis vector pa, a path from i to l.

For example, if Q is the Kronecker quiver

$$2 \underset{b}{\overset{a}{\rightrightarrows}} 1$$

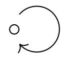
then $P_1 = S_1$ is the representation

$$0 \underset{0}{\overset{0}{\rightrightarrows}} k$$

and P_2 is the representation

$$k \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} k^2$$

If Q is the Jordan quiver



Then $kQ = k[t]$, the indecomposable projective is $P = k[t]$ itself, and the simple module is $S = k[t]/(t)$, which corresponds to the representation $(k, 0)$. On the other hand, the representations $S_\lambda := (k, \lambda)$ for $\lambda \in k$ are pairwise non-isomorphic simples. Note that the corresponding module is $k[t]/(t - \lambda)$.

6.6 Finite dimensional hereditary algebras

We have described the indecomposable projective modules and certain simple modules for the hereditary tensor algebras. In this section we show that a similar description also holds for all finite dimensional hereditary algebras (or more generally hereditary Artinian algebras, or even hereditary semiprimary algebras). In the process we will prove that every such hereditary algebra R can be written as $R = A \oplus J$ where J is a nilpotent ideal (the Jacobson radical) and A is a semisimple subalgebra.

Let R be an algebra. We define its Jacobson radical to be

$$\mathrm{Jac}(R) := \bigcap_{S \text{ simple}} \mathrm{Ann}_R(S) = \{x \in R : Sx = 0 \text{ for all simples } S\}.$$

We say that R is Artinian if every descending chain of ideals stabilises. Clearly, if the regular module has finite length (for example if the algebra is finite dimensional over a field), then it must be Artinian. The next result shows the converse also holds.

Theorem 6.15. *An algebra R is Artinian if and only if the regular module has finite length, in which case the Jacobson radical J is nilpotent and R/J is a semisimple algebra.*

Proof. We begin by observing that if S is a simple module, then its annihilator $\mathrm{Ann}_R(S) := \{x \in R : Sx = 0\}$ is a two-sided ideal, so J is a two-sided ideal. On the other hand, the annihilator can also be written as the intersection $\bigcap_{0 \neq s \in S} I_s$, where $I_s = \{x \in R : sx = 0\}$. We observe that I_s is the kernel of the surjective map $R \to S$, $1 \mapsto s$, so I_s is a maximal right ideal. Thus J is also an intersection of maximal right ideals.

Now, we have already noted that finite length implies Artinian, so assume that R is Artinian. We first show that J is a nilpotent ideal. The descending chain $J \supset J^2 \supset J^3 \supset \cdots$ must stabilise, so $J^n = J^{n+1}$ for some n. Set $I := \{r : rJ^n = 0\}$, a two-sided ideal of R. Assume for contradiction that $I \neq R$. Then R/I is again an Artinian algebra, so must contain a minimal right ideal, necessarily of the form L/I for some right ideal L of R. Now L/I is a simple R-module, so $(L/I)J = 0$, or in other words, $LJ \subset I$. Then

$$LJ^n = LJ^{n+1} \subset IJ^n = 0,$$

so $L \subset I$, giving the required contradiction. Thus $I = R$ and so $J^n = 0$.

Next, we saw above that J is an intersection of maximal right ideals. Since R is Artinian, it must be an intersection of finitely many maximal right ideals, say $I_1, \ldots, I_n$. Set $S_i := R/I_i$, a simple R-module. Then J is the kernel of the natural map $R \to \bigoplus_i S_i$, so R/J is a submodule of the semisimple module $\bigoplus_i S_i$, hence is itself semisimple. Thus R/J is a semisimple algebra.

To see this, take $C \subset \{1, \ldots, n\}$ maximal such that $(R/J) \cap \bigoplus_{i \in C} S_i = 0$. Set $X := (R/J) \oplus \bigoplus_{i \in C} S_i$. Then for each j, the intersection $S_j \cap X$ is a submodule of the simple S_j, so is either 0 or S_j itself. If it is 0 for some j, then we could form the direct sum $X \oplus S_j$, and hence could replace C by $C \cup \{j\}$, a contradiction. Thus every S_j is contained in X, so $X = \bigoplus_i S_i$. It follows from the Krull-Remak-Schmidt Theorem that $R/J \cong \bigoplus_{i \notin C} S_i$, so is semisimple.

Finally, each subquotient J^r/J^{r+1} is an R/J-module, so is a direct sum of simple modules, and since R is Artinian, it must necessarily be a finite direct sum, so has finite length. Using that J is nilpotent, we conclude that the regular module itself has finite length. □

Aside. In general, we say that R is semiprimary if its Jacobson radical J is nilpotent and R/J is a semisimple algebra. Thus Artinian implies semiprimary, whereas a semiprimary algebra is Artinian if and only if J/J^2 has finite length. Everything? in this section concerning hereditary Artinian algebras holds more generally for semiprimary algebras.

Lemma 6.16. *Let R be hereditary, and P and P' projective R-modules with P indecomposable. Then any non-zero homomorphism $P \to P'$ is a monomorphism. In particular, if P has finite length, then $\mathrm{End}_R(P)$ is a division algebra. If R is Artinian, then every indecomposable projective is isomorphic to a direct summand of R, and hence has finite length.*

Proof. Let $f \colon P \to P'$ be non-zero. Then $\mathrm{Im}(f) \leq P'$ is projective, so $P \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$, and since P is indecomposable and f is non-zero, we must have $\mathrm{Ker}(f) = 0$.

Now suppose that P has finite length. By Fitting's Lemma, every non-invertible endomorphism is nilpotent, so must be zero. Hence $\mathrm{End}_R(P)$ is a division algebra.

Finally, suppose that R is Artinian. Then P is a direct summand of a free module $R^{(I)}$, then some projection $P \to R^{(I)} \to R$ is non-zero, hence is injective, and so P has finite length. Thus we have an epimorphism $R^n \twoheadrightarrow P$, so P is isomorphic to a direct summand of R^n, and hence by the Krull-Remak-Schmidt Theorem, P is isomorphic to an indecomposable summand of R. □

Remark. It is true that every finitely generated indecomposable projective module for an hereditary tensor algebra $T_A(M)$ is isomorphic to some P_i. There exists a proof for path algebras of quivers using Gröbner bases, and a somewhat obscure proof in general. Would like to find a better proof...

In general there may be many indecomposable projective modules. For a Dedekind domain R we have $K_0(R) \cong \mathbb{Z} \oplus \mathrm{Cl}(R)$, where $\mathrm{Cl}(R)$ is the (divisor) class group, which is is trivial if and only if R is a principal ideal domain.

Theorem 6.17. *Let R be an Artinian algebra, and decompose the regular module $R = \bigoplus_i P_i$ into a direct sum of indecomposable projective modules. Write $P_i = \varepsilon_i R$ with $\varepsilon_i \in R$ a primitive idempotent. Then*

1. *$S_i := P_i/P_iJ$ is a simple module, and every simple R-module is isomorphic to some S_i.*
2. *Setting $(J/J^2)^\vee := \mathrm{Hom}_{R/J}(J/J^2, R/J)$ we have*
$$\mathrm{Ext}^1_R(S_i, S_j) \cong \varepsilon_j (J/J^2)^\vee \varepsilon_i.$$
3. *Assume further that R is hereditary. Then $\mathrm{End}_R(P_i) \cong \mathrm{End}_R(S_i)$ and $R = A \oplus J$, where A is a semisimple subalgebra of R.*

Proof. (1) Write $\pi_i \colon P_i \to S_i$ for the canonical epimorphism. Then there is a natural algebra map $\mathrm{End}_R(P_i) \to \mathrm{End}_R(S_i)$, which sends f to the unique $\bar{f}$ such that $\bar{f}\pi_i = \pi_j f$. It is surjective, since given any $g \colon S_i \to S_j$, we can use that P_i is projective to lift the map $g\pi_i$ along π_j.

We know that P_i has finite length, so by Fitting's Lemma its endomorphism algebra is a local algebra. Thus $\mathrm{End}_R(S_i)$ is a local algebra, so again by Fitting's Lemma, S_i is indecomposable. On the other hand, we know that S_i is an R/J-module, so is semisimple. Hence S_i is simple.

Now let S be any simple module. Then we have an epimorphism $R \twoheadrightarrow S$, whose kernel necessarily contains J, so yields an epimorphism $R/J \twoheadrightarrow S$. Since $R/J = \bigoplus_i S_i$, some $S_i \to S$ is non-zero, and hence an isomorphism.

(2) We follow the proof used for tensor algebras. We have $SJ = 0$ for all simple modules S, so for any R-module X we have
$$\mathrm{Hom}_R(X, S) \cong \mathrm{Hom}_R(X/XJ, S) \cong \mathrm{Hom}_{R/J}(X/XJ, S).$$
In particular, applying $\mathrm{Hom}_R(-, S_j)$ to the short exact sequence $0 \to P_iJ \to P_i \to S_i \to 0$ yields
$$\mathrm{Ext}^1_R(S_i, S_j) \cong \mathrm{Hom}_R(P_iJ, S_j) \cong \mathrm{Hom}_{R/J}(\varepsilon_i(J/J^2), \varepsilon_j(R/J)) \cong \varepsilon_j(J/J^2)^\vee \varepsilon_i.$$

(3) Fix representatives $P_1, \ldots, P_n$ for the isomorphism classes of indecomposable projectives and write $R = \bigoplus_j P'_j$, where $P'_j \cong P_j^{d_j}$. Write $P'_j = \varepsilon'_j R$ for some idempotents ε'_j, so that $R = \sum_{i,j} \varepsilon'_j R \varepsilon'_i$.

Now $\varepsilon'_j R \varepsilon'_i \cong \mathrm{Hom}_R(P'_i, P'_j)$. If $i = j$, then this is isomorphic to the semisimple algebra $\mathbb{M}_{d_i}(\mathrm{End}_R(P_i))$. Thus $A := \bigoplus_i \varepsilon'_i R \varepsilon'_i$ is a semisimple subalgebra of R.

On the other hand, if $i \neq j$, then every homomorphism between the indecomposable projectives $P_i \to P_j$ has image contained in P_jJ. Thus every homomorphism $P'_i \to P'_j$ has image contained in P'_jJ. So, given $x \in \varepsilon'_j R \varepsilon'_i$, it corresponds to the homomorphism $f\colon P_i' \to P_j'$, $f(\varepsilon_i') = x$. Since f has image contained in $P_j'J$, we must have $x \in \varepsilon_j'J$. We conclude that $\varepsilon_j'R\varepsilon_i' = \varepsilon_j'J\varepsilon_i'$. Thus $J = \bigoplus_{i\neq j} \varepsilon_j'R\varepsilon_i'$. □

Theorem 6.18. *Let R be hereditary Artinian, and let $P_1, \ldots, P_n$ be representatives for the isomorphism classes of indecomposable projective modules. Then, up to reordering, we may assume that $\mathrm{Hom}_R(P_j, P_i) = 0$ for $i < j$. (This says that R is a triangular algebra.)*

It follows that $R = A \oplus J$, where A is a semisimple subalgebra and J is the Jacobson radical of R.

Proof. Suppose $\mathrm{Hom}_R(P_i, P_j) \neq 0$. Then we have a monomorphism $P_i \rightarrowtail P_j$, and since this is not an isomorphism, it is not onto. Thus $\ell(P_i) < \ell(P_j)$, and so $\mathrm{Hom}_R(P_j, P_i) = 0$. It follows that, up to reordering, we have $\mathrm{Hom}_R(P_j, P_i) = 0$ for all $i < j$.

Now write $R = \bigoplus_i P_i'$, where $P_i' \cong P_i^{d_i}$. Write $P_i' = \varepsilon_i'R$ for some idempotents ε_i'. Since $\varepsilon_i'R\varepsilon_j' \cong \mathrm{Hom}_R(P_j', P_i') = 0$ for all $i < j$, we have

$$R = \bigoplus_{i,j} \varepsilon_j'R\varepsilon_i' = A \oplus J',$$

where $A := \bigoplus_i \varepsilon_i'R\varepsilon_i'$ is a subalgebra of R and $J' := \bigoplus_{i<j} \varepsilon_j'R\varepsilon_i'$ is a two-sided nilpotent ideal. By the earlier lemma, each $\mathrm{End}_R(P_i)$ is a division algebra, so

$$\varepsilon_i'R\varepsilon_i' \cong \mathrm{End}_R(P_i') \cong \mathrm{End}_R(P_i^{d_i}) \cong \mathbb{M}_{d_i}(\mathrm{End}_R(P_i))$$

is a semisimple algebra, so A is a semisimple subalgebra of R. In particular $R/J' \cong A$ is a semisimple R-module, so $(R/J')J = 0$, or in other words, $J \subset J'$. Conversely, let S be any simple R-module. If $SJ' = S$, then since J' is nilpotent, we must have $S = SJ' = S(J')^2 = \cdots = 0$, a contradiction. Thus $SJ' = 0$, and since this holds for all simples S, we see that $J' \subset J$.

Hence $J' = J$ and $R = A \oplus J$. □

With this convention, we label the vertices of a quiver without oriented cycles by starting at the sinks, and finishing at the sources.

We finish this section by showing that the centre of an hereditary Artinian algebra is a product of fields. In particular, if R is an indecomposable algebra (so has no central idempotents other than 0 or 1), then its centre is a field. Thus we may always assume that we have an hereditary Artinian k-algebra over some field k. (Note that R may still not be finite dimensional over k.)

Theorem 6.19. *Let R be an hereditary Artinian algebra. Then its centre Z is a product of fields.*

Proof. Decompose $R = \bigoplus_i P_i$ as a direct sum of indecomposable projectives. Take $0 \neq z \in Z$. Then z induces an endomorphism of each P_i, and every non-zero endomorphism is an automorphism. Hence we can find an idempotent $e \in R$ such that $z = ez$, and multiplication by z is an automorphism of eR. In particular, $e = zx$ for some $x \in eR$. We claim that e and x are both central.

Write $R = eR \oplus e'R$, where $e' = 1-e$. Then $e'z = 0$ and multiplication by z is an automorphism of eR. Thus $e'R = \{r : rz = 0\}$, and hence $eRe' = 0 = e'Re$. For, take $r \in eRe'$. Then $rz = 0$, so $r = er = xzr = xrz = 0$. Similarly if $r \in e'Re$. Hence $R = eRe \oplus e'Re'$.

Next

$$er + e'r = r = re + re' \quad \text{so} \quad er - re = re' - e'r = 0,$$

so e is central. Then, for all $r \in R$, we have $rx - xr \in \varepsilon R$, but also $z(rx - xr) = re - er = 0$, so $rx - xr \in e'R$. Thus $rx - xr = 0$, and hence x is central.

Finally, write $Z = \bigoplus_j e_j Z$ as a direct sum of indecomposable projective Z-modules. Then e_j is the only non-zero idempotent in $e_j Z \cong \mathrm{End}_Z(e_j Z)$. So, if $z \in e_j Z$ is non-zero, then by the above zx is a non-zero idempotent for some $x \in Z$, so $zx = e_j$ and hence $e_j Z$ is a field (with unit e_j). □

An algebra R is said to be basic if the regular module is a direct sum of pairwise non-isomorphic indecomposable projective modules. It is known that every algebra is Morita equivalent to a basic one. For $R = A \oplus J$ hereditary, where A is semisimple and $\bigcap_n J^n = 0$, we see that R is basic if and only if A is basic, which is if and only if A is a product of division algebras (all matrices have size one).

7 Finite representation type

7.1 The Grothendieck group

Let k be a field, and R an hereditary k-algebra. Assume further that R is either finite dimensional over k, so $R = A \oplus J$ where J is the Jacobson radical and A is a semisimple subalgebra, and we set $M := J/J^2$, or else $R = T_A(M)$ is a tensor algebra such that both A and M are finite dimensional over k. Finally we will assume that A is basic, so $A = A_1 \times \cdots \times A_n$ with each A_i a division algebra. Write $1 = \sum_i \varepsilon_i$, where ε_i is the unit in A_i.

We consider the subcategory of R-modules consisting only of those which are finite dimensional over k. It is clear that subquotients of and extensions of such modules remain finite dimensional over k, so they form a nice (Serre) subcategory.

The Grothendieck group of A is the free abelian group Γ with basis the isomorphism classes of simple A-modules. We write $[X]$ for the isomorphism class of an A-module X, so that $\Gamma \cong \mathbb{Z}^n$ with basis $e_i := [S_i]$, where $S_i = A_i = \varepsilon_i A$. Now, using that A is a subalgebra of R, if X is an R-module, then we can restrict attention to A to obtain an A-module X_A; moreover, if X is finite dimensional over k, then X_A decomposes as a finite direct sum of simple A-modules, and so we have its class $[X_A] \in \Gamma$.

We now equip Γ with a bilinear form, called the Euler form,

$$\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}, \quad \langle e_i, e_j \rangle := \delta_{ij} \dim_k A_i - \dim_k \varepsilon_i M \varepsilon_j.$$

Example 7.1. *Let $R = kQ$ be the path algebra of a quiver. Then $\Gamma \cong \mathbb{Z}^n$ has basis e_i indexed by the vertices of Q, and every finite dimensional representation $X = (X_i, \xi_a)$ is sent to its dimension vector $\underline{\dim}\, X := \sum_i (\dim X_i) e_i$.*

Moreover, the bilinear form is determined by $\langle e_i, e_j \rangle := \delta_{ij} - a_{ij}$, where a_{ij} is the number of arrows from i to j.

Proposition 7.2. *Let X and Y be finite dimension R-modules. Then both $\operatorname{Hom}_R(X, Y)$ and $\operatorname{Ext}^1_R(X, Y)$ are finite dimensional, and we have*

$$\langle X, Y \rangle = \dim_k \operatorname{Hom}_R(X, Y) - \dim_k \operatorname{Ext}^1_k(X, Y).$$

Proof. We set $\{X, Y\} := \dim_k \operatorname{Hom}_R(X, Y) - \dim_k \operatorname{Ext}^1_R(X, Y)$. We need to check that this is finite, depends only on the classes of X_A and Y_A, and that it equals $\langle X_A, Y_A \rangle$.

Suppose first that $R = T_A(M)$ is a tensor algebra. Then we can apply $\operatorname{Hom}_R(-, Y)$ to the standard exact sequence

$$0 \to X \otimes_A M \otimes_A R \to X \otimes_A R \to X \to 0$$

to obtain the four term exact sequence

$$0 \to \operatorname{Hom}_R(X, Y) \to \operatorname{Hom}_A(X, Y) \to \operatorname{Hom}_A(X \otimes_A M, Y) \to \operatorname{Ext}^1_R(X, Y) \to 0.$$

The first three terms are finite dimensional, so the fourth is as well, and

$$\{X, Y\} = \dim_k \operatorname{Hom}_A(X, Y) - \dim_k \operatorname{Hom}_A(X \otimes_A M, Y),$$

which only depends on the classes of X_A and Y_A. We now compute the right hand side when $X = S_i$ and $Y = S_j$. Now $\operatorname{Hom}_A(S_i, S_j) = 0$ unless $i =$

j, in which case it is isomorphic to A_i, so $\dim_k \operatorname{Hom}_A(S_i, S_j) = \delta_{ij} \dim_k A_i$. Also, $\operatorname{Hom}_A(S_i \otimes_A M, S_j) \cong \operatorname{Hom}_{A_j}(\varepsilon_i M \varepsilon_j, A_j)$. Since A_j is a division algebra, every right module is free, so we can write $\varepsilon_i M \varepsilon_j \cong A_j^d$ for some d. Then $\operatorname{Hom}_{A_j}(\varepsilon_i M \varepsilon_j, A_j) \cong A_j^d$, and so computing dimensions over k we have

$$\dim_k \operatorname{Hom}_A(\varepsilon_i M \varepsilon_j, A_j) = d \dim_k A_j = \dim_k \varepsilon_i M \varepsilon_j.$$

Suppose instead that R is finite dimensional, so that every simple R-module is isomorphic to some S_i. Given a short exact sequence

$$0 \to X' \to X \to X'' \to 0,$$

we can apply $\operatorname{Hom}_R(-, Y)$ to obtain a six term exact sequence

$$\begin{array}{l} 0 \longrightarrow \operatorname{Hom}_R(X'', Y) \longrightarrow \operatorname{Hom}_R(X, Y) \longrightarrow \operatorname{Hom}_R(X', Y) \\ \quad \longrightarrow \operatorname{Ext}^1_R(X'', Y) \longrightarrow \operatorname{Ext}^1_R(X, Y) \longrightarrow \operatorname{Ext}^1_R(X', Y) \longrightarrow 0. \end{array}$$

Taking dimensions we obtain

$$\{X, Y\} = \{X', Y\} + \{X'', Y\}.$$

Similarly, if $0 \to Y' \to Y \to Y'' \to 0$ is a short exact sequence, then $\{X, Y\} = \{X, Y'\} + \{X, Y''\}$. It follows that $\{-, -\}$ depends only on the simple composition factors of both X and Y, so we may assume that they are both simple. As above, $\dim_k \operatorname{Hom}_R(S_i, S_j) = \delta_{ij} \dim_k A_i$, and

$$\dim_k \operatorname{Ext}^1_R(S_i, S_j) = \dim_k \operatorname{Hom}_{A_j}(\varepsilon_i M \varepsilon_j, A_j) = \dim_k \varepsilon_i M \varepsilon_j,$$

where now $M = J/J^2$.

This proves that, in both cases, $\{X, Y\} = \langle X_A, Y_A \rangle$ for all finite dimensional R-modules X and Y. □

The form $\langle -, - \rangle$ on Γ is bilinear, but not symmetric. We therefore define the following symmetric bilinear form on Γ

$$(X, Y) := \langle X, Y \rangle + \langle Y, X \rangle.$$

Proposition 7.3. *Let R be a finite dimensional hereditary k-algebra. Then the Euler form $\langle -, - \rangle$ is non-degenerate.*

Proof. We know that R is triangular, so we may assume that $\varepsilon_i J \varepsilon_j = 0$ for $i < j$. Thus the matrix representing the bilinear form $\langle -, - \rangle$ is lower triangular with the numbers $\dim_k A_i$ on the diagonal, and hence is an invertible matrix. Thus $\langle x, y \rangle = 0$ for all y implies $x = 0$, and dually $\langle x, y \rangle = 0$ for all x implies $y = 0$. Thus the form is non-degenerate. □

This result can fail for quivers. For example, the bilinear form for the Jordan quiver is identically zero. It can also fail if we replace the non-symmetric bilinear form by the symmetric bilinear. For, the non-symmetric bilinear form on the Kronecker quiver is given by $\left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)$, which is non-degenerate, so the symmetric bilinear form is given by $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, which is degenerate.

7.2 Some natural isomorphisms

Let k be a field, R a k-algebra, and $D = \mathrm{Hom}_k(-,k)$ the usual vector space duality. Thus, if X is a right R-module, then DX is a left R-module, via $(rf)(x) := f(xr)$ for $f \in DX$, $r \in R$ and $x \in X$. Similarly, if Y is a left R-module, then DY is a right R-module, via $(fr)(y) := f(ry)$ for $f \in DY$, $r \in R$ and $y \in Y$.

Lemma 7.4. *Let X be a right R-module, and Y a left R-module. Then we have natural isomorphisms*

$$\mathrm{Hom}_R(X, DY) \cong D(X \otimes_R Y) \cong \mathrm{Hom}_R(Y, DX).$$

Proof. These follow from the usual tensor-hom adjunction. For example

$$\mathrm{Hom}_R(X, DY) = \mathrm{Hom}_R(X, \mathrm{Hom}_k(Y, k)) \cong \mathrm{Hom}_k(X \otimes_R Y, k) = D(X \otimes_R Y). \quad \square$$

Our next two results require a little more category theory. Recall that we have the (abelian) category of right R-modules, denoted $\mathrm{Mod}\, R$. An (additive) functor $F\colon \mathrm{Mod}\, R \to \mathrm{Mod}\, k$ is an assignment of a vector space $F(X)$ for each R-module X, and a linear map $\mathrm{Hom}_R(X,Y) \to \mathrm{Hom}_k(F(X), F(Y))$, $f \mapsto F(f)$, such that $F(gf) = F(g)F(f)$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$. Examples include $\mathrm{Hom}_R(-,Z)$ or $\mathrm{Ext}^1_R(-,Z)$ for some fixed right R-module Z, or $- \otimes_R Z$ for some fixed left R-module Z.

Given two such functors F and G, a natural transformation $\eta\colon F \to G$ consists of a linear map $\eta_X\colon F(X) \to G(X)$ for all R-modules X, such that for all $f\colon X \to Y$ we have the commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y). \end{array}$$

We say that η is a natural isomorphism provided η_X is an isomorphism for all X.

Lemma 7.5. *Suppose we have a natural transformation $\eta\colon F \to G$ between two functors $F, G\colon \mathrm{Mod}\, R \to \mathrm{Mod}\, k$. If η_R is an isomorphism, then η_P is an isomorphism for all finitely generated projective modules P.*

Proof. We begin by observing that there is an isomorphism

$$\begin{pmatrix} F(p_X) \\ F(p_Y) \end{pmatrix} \colon F(X \oplus Y) \xrightarrow{\sim} F(X) \oplus F(Y),$$

with inverse $(F(i_X), F(i_Y))$. Thus the natural transformation $\eta_{X \oplus Y}$ yields a linear map $\theta\colon F(X) \oplus F(Y) \to G(X) \oplus G(Y)$, which we can regard as a matrix. For example the component $F(X) \to G(X)$ is given by

$$G(p_X)\eta_{X\oplus Y}F(i_X) = \eta_X F(p_X)F(i_X) = \eta_X F(\mathrm{id}_X) = \eta_X,$$

using the commutativity for the homomorphism p_X. Similarly, the component $F(X) \to G(Y)$ is given by

$$G(p_Y)\eta_{X\oplus Y}F(i_X) = \eta_Y F(p_Y)F(i_X) = \eta_Y F(0) = 0.$$

Thus the induced linear map is precisely $\theta = \eta_X \oplus \eta_Y$.

So, $\eta_{X\oplus Y}$ is an isomoprhism if and only if both η_X and η_Y are isomorphisms. Now, if η_R is an isomorphism, then so too is η_{R^n}, and hence also η_P for any direct summand P of R^n, so for any finitely generated projective R-module P. $\square$

Lemma 7.6. *We have the following natural isomorphisms, for all finitely generated projective right R-modules P, all right R-modules X, and all left R-modules Y.*

1. $X \otimes_R \mathrm{Hom}_R(P,R) \cong \mathrm{Hom}_R(P,X)$.
2. $P \otimes_R Y \cong \mathrm{Hom}_R(\mathrm{Hom}_R(P,R),Y)$.

Proof. By the lemma above, we need to construct in each case a natural transformation of functors, and show that it is an isomorphism when evaluated at R.

1. Consider the map $\eta_Z \colon X \otimes_R \mathrm{Hom}_R(Z,R) \to \mathrm{Hom}_R(Z,X)$ sending $x\otimes f$ to the homomorphism $z \mapsto xf(z)$. This is a natural transformation of functors, and η_R is an isomorphism.

2. Consider the map $\eta_Z \colon Z \otimes_R Y \to \mathrm{Hom}_R(\mathrm{Hom}_R(Z,R),Y)$ sending $z\otimes y$ to the homomorphism $f \mapsto f(z)y$. This is a natural transformation of functors, and η_R is an isomorphism.

Alternatively, we can put $Y = R$ into (2) to get $\mathrm{Hom}_R(\mathrm{Hom}_R(P,R),R) \cong P \otimes_R R \cong P$. Then (2) also yields $\mathrm{Hom}_R(P',R) \otimes_R Y \cong \mathrm{Hom}_R(P',Y)$ for all finitely generated projective left R-modules P', and swapping left and right yields (1). $\square$

7.3 The Auslander-Reiten translate

Let k be a field, and $R = A \oplus J$ a finite dimensional hereditary k-algebra. Write $D := \mathrm{Hom}_k(-,k)$ for the usual vector space duality. We define the Auslander-Reiten translate τ and the inverse translate τ^- on finite dimensional R-modules via

$$\tau X := D\,\mathrm{Ext}^1_R(X,R) \quad \text{and} \quad \tau^- X := \mathrm{Ext}^1_R(DR,X).$$

Lemma 7.7. *If X is a finite dimensional right R-module, then so too are τX and $\tau^- X$.*

Proof. Let X be a finite dimensional right R-module. Since R, and hence also DR, is finite dimensional, we know from Proposition 7.2 that both $\mathrm{Ext}^1_R(X,R)$ and $\mathrm{Ext}^1_R(DR,X)$ are finite dimensional, so τX and $\tau^- X$ are both finite dimensional.

Next, $\mathrm{Ext}^1_R(X,R)$ is naturally a left module over $\mathrm{End}_R(R) \cong R$ via the push-out map. Explicitly, given a short exact sequence $0 \to R \to E \to X \to 0$ and an element $r \in R$, we can push-out along the map $R \to R$, $s \mapsto rs$. Thus $\tau X := D\,\mathrm{Ext}^1_R(X,R)$ is naturally a right R-module.

Dually, $\mathrm{Ext}^1_R(DR,X)$ is a finite dimensional right $\mathrm{End}_R(DR)$-module via the pull-back map. Then, as in the previous section, and using that R is finite dimensional, we have

$$\mathrm{End}_R(DR) \cong D(DR \otimes_R R) \cong D^2(R) \cong R.$$

Moreover, this vector space isomorphism is in fact an algebra isomorphism. Thus $\tau^- X := \operatorname{Ext}^1_R(DR, X)$ is naturally a right R-module via the pull-back map. ☐

Lemma 7.8. *Given a homomorphism $f\colon X \to Y$, the push-out along f yields a homomorphism $\tau^- X \to \tau^- Y$, and the pull-back along f induces a homomorphism $\tau X \to \tau Y$. (In other words, $\tau^\pm$ are functors.)*

Proof. We have already seen that the push-out along f yields a homomorphism $\operatorname{Ext}^1_R(DR, X) \to \operatorname{Ext}^1(DR, Y)$ of right R-modules. Similarly, pull-back along f yields a homomorphism $\operatorname{Ext}^1_R(Y, R) \to \operatorname{Ext}^1_R(X, R)$ of left R-modules, so taking duals gives a homomorphism $D\operatorname{Ext}^1_R(X, R) \to D\operatorname{Ext}^1_R(Y, R)$. ☐

Theorem 7.9 (Auslander-Reiten Formula)**.** *Let X and Y be finite dimensional R-modules. Then we have natural isomorphisms*

$$\operatorname{Hom}_R(\tau^- X, Y) \cong D\operatorname{Ext}^1_R(Y, X) \cong \operatorname{Hom}_R(X, \tau Y).$$

(In other words, (τ^-, τ) form an adjoint pair.)

Proof. Take a short exact sequence $0 \to P_1 \xrightarrow{g} P_0 \to Y \to 0$ with P_0 (and hence also P_1) a finite dimensional projective R-module; for example we could take $P_0 = Y \otimes_A R$. Now consider the two functors

$$F := D\operatorname{Hom}_R(-, X) \quad \text{and} \quad G := \operatorname{Hom}_R(X, D\operatorname{Hom}_R(-, R))$$

and the natural transformation $\eta\colon F \to G$ as in Lemma 7.6 (1)

$$\eta_Z\colon D\operatorname{Hom}_R(Z, X) \to D(X \otimes_R \operatorname{Hom}_R(Z, R)) \xrightarrow{\sim} \operatorname{Hom}_R(X, D\operatorname{Hom}_R(Z, R)),$$

which is an isomorphism whenever Z is a finite dimensional projective R-module.

We now apply this to the homomorphism $g\colon P_1 \to P_0$ between finite dimensional projective modules to obtain the following commutative diagram where the horizontal maps are both isomorphisms

$$\begin{array}{ccc} F(P_1) & \xrightarrow[\sim]{\eta_{P_1}} & G(P_1) \\ \downarrow{\scriptstyle F(g)} & & \downarrow{\scriptstyle G(g)} \\ F(P_0) & \xrightarrow[\sim]{\eta_{P_0}} & G(P_0). \end{array}$$

This induces a (natural) isomorphism between the kernels of the vertical maps

$$D\operatorname{Ext}^1_R(Y, X) \xrightarrow{\sim} \operatorname{Hom}_R(X, \tau Y).$$

For the second isomorphism, we will do slightly more. This time we start with a short exact sequence $0 \to X \to I_0 \xrightarrow{f} I_1 \to 0$ with I_0 (and hence also I_1) a finite dimensional injective R-module. Note that such a sequence always exists. For, we can a short exact sequence $0 \to P_1' \to P_0' \to DX \to 0$ with P_0' a finite dimensional projective left R-module and then apply the vector space duality D.

We now introduce the following four functors

$$F := D\operatorname{Hom}_R(Y, -) \quad \text{and} \quad G := \operatorname{Hom}_R(\operatorname{Hom}_R(DR, -), Y)$$

and also

$$\tilde{F} := Y \otimes_R D(-) \quad \text{and} \quad \tilde{G} := \mathrm{Hom}_R(\mathrm{Hom}_R(D(-), R), Y)$$

Again, as in the previous section we have natural isomorphisms

$$F(Z) \xrightarrow{\sim} \tilde{F}(Z) \quad \text{and} \quad \tilde{G}(Z) \xrightarrow{\sim} G(Z),$$

(where we have used that R is finite dimensional, so $R \cong D^2R$), as well as a natural transformation

$$\tilde{F}(Z) = Y \otimes_R DZ \to \mathrm{Hom}_R(\mathrm{Hom}_R(DZ, R), Y) = \tilde{G}(Z)$$

which is an isomorphism whenever DZ is a finite dimensional projective (left) R-module, equivalently Z is a finite dimensional injective right R-module.

Applying these functors to the homomorphism $f\colon I_0 \to I_1$ we obtain the commutative diagram where all horizontal maps are isomorphisms

$$\begin{array}{ccccccc} F(I_1) & \xrightarrow{\sim} & \tilde{F}(I_1) & \xrightarrow{\sim} & \tilde{G}(I_1) & \xrightarrow{\sim} & G(I_1) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle \tilde{F}(f)} & & \downarrow{\scriptstyle \tilde{G}(f)} & & \downarrow{\scriptstyle G(f)} \\ F(I_0) & \xrightarrow{\sim} & \tilde{F}(I_0) & \xrightarrow{\sim} & \tilde{G}(I_0) & \xrightarrow{\sim} & G(I_0). \end{array}$$

This yields (natural) isomorphisms between the kernels of the columns

$$D\,\mathrm{Ext}^1_R(Y, X) \xrightarrow{\sim} \mathrm{Ker}(\tilde{F}(f)) \xrightarrow{\sim} \mathrm{Ker}(\tilde{G}(f)) \xrightarrow{\sim} \mathrm{Hom}_R(\tau^- X, Y).$$

We finish by observing that $\mathrm{Ker}(\tilde{G}(f)) = \mathrm{Hom}_R(\mathrm{Ext}^1_R(DX, R), Y)$. We also have $\mathrm{Ker}(\tilde{F}(f)) = \mathrm{Tor}^R_1(Y, DX)$, though we have not instroduced the Tor functors. □

Corollary 7.10. *The inverse translate* $\tau^- X := \mathrm{Ext}^1_R(DR, X)$ *is naturally isomorphic to the functor* $\mathrm{Ext}^1_R(DX, R)$.

Proof. In the proof of the theorem we used that there is a natural isomorphism $\mathrm{Hom}_R(DR, -) \cong \mathrm{Hom}_R(D(-), R)$. So, if we have a short exact sequence $0 \to X \to I_0 \xrightarrow{g} I_1 \to 0$ with I_0 finite dimensional and injective, then we can apply the functors to the map g to obtain a (natural) isomorphism between their cokernels $\mathrm{Ext}^1_R(DR, X) \cong \mathrm{Ext}^1_R(DX, R)$. □

7.4 First properties

Theorem 7.11. *A module* X *is projective if and only if* $\tau X = 0$. *On the other hand, if* X *has no projective summands, then* $\tau^- \tau X \cong X$.

Dually, a module Y *is injective if and only if* $\tau^- Y = 0$. *On the other hand, if* Y *has no injective summands, then* $\tau\tau^- Y \cong Y$.

Proof. Suppose X is projective. Then by definition $\tau X = D\,\mathrm{Ext}^1_R(X, R) = 0$. Suppose instead that $\tau X = 0$. Then by the Auslander-Reiten Formula, $\mathrm{Ext}^1_R(X, Y) \cong D\,\mathrm{Hom}_R(Y, \tau X) = 0$. This holds for all finite dimensional Y, which since R and X are both finite dimensional, is enough to prove that X is projective.

Now assume X contains no projective summands, and take a short exact sequence $0 \to P_1 \xrightarrow{f} P_0 \to X \to 0$ with P_0 finite dimensional projective. By Lemma 7.6 (2) we have a natural transformation

$$\eta_Z\colon Z \cong Z \otimes_R R \to \mathrm{Hom}_R(\mathrm{Hom}_R(Z, R), R)$$

which is an isomorphism whenever Z is a finite dimensional projective module. So, applying this to the homomorphism $f\colon P_1 \to P_0$, we obtain a commutative square where the horizontal maps are isomorphisms

$$\begin{array}{ccc} P_1 & \xrightarrow[\sim]{\eta_{P_1}} & \mathrm{Hom}_R(\mathrm{Hom}_R(P_1, R), R) \\ \big\downarrow f & & \big\downarrow (f^*)^* \\ P_0 & \xrightarrow[\sim]{\eta_{P_0}} & \mathrm{Hom}_R(\mathrm{Hom}_R(P_0, R), R) \end{array}$$

This therefore induces a (natural) isomorphism between the cokernels of the vertical maps. Clearly the cokernel of the left hand map is X, so we just need to compute the cokernel of the right hand map.

Since R is hereditary, our assumption that X contains no projective summands implies that $\mathrm{Hom}_R(X, R) = 0$. Thus applying $\mathrm{Hom}_R(-, R)$ yields the short exact sequence of left R-modules

$$0 \to \mathrm{Hom}_R(P_0, R) \xrightarrow{f^*} \mathrm{Hom}_R(P_1, R) \to \mathrm{Ext}^1_R(X, R) = D\tau X \to 0.$$

We next observe that the functor $\mathrm{Hom}_R(-, R)$ sends finite dimensional projective right R-modules to finite dimensional projective left R-modules. For, suppose $P \oplus Q \cong R^n$. Then we have isomorphisms of left R-modules

$$\mathrm{Hom}_R(P, R) \oplus \mathrm{Hom}_R(Q, R) \cong \mathrm{Hom}_R(P \oplus Q, R) \cong \mathrm{Hom}_R(R^n, R) \cong R^n.$$

It follows that the sequence above is a projective resolution of $D\tau X$. Applying $\mathrm{Hom}_R(-, R)$ again proves that the cokernel of $(f^*)^*$ is $\mathrm{Ext}^1_R(D\tau X, R)$, which is isomorphic to $\tau^-\tau X$ by Corollary 7.10. □

Corollary 7.12. *Suppose X has no projective summands. Then*

$$\mathrm{End}_R(\tau X) \cong \mathrm{End}_R(X) \quad \text{and} \quad \mathrm{Ext}^1_R(\tau X, \tau X) \cong \mathrm{Ext}^1_R(X, X).$$

Dually, if X has no injective summands. then

$$\mathrm{End}_R(\tau^- X) \cong \mathrm{End}_R(X) \quad \text{and} \quad \mathrm{Ext}^1_R(\tau^- X, \tau^- X) \cong \mathrm{Ext}^1_R(X, X).$$

Proof. These follow from the Auslander-Reiten Formula, together with the (natural) isomorphism $\tau^-\tau X \cong X$. For, we have

$$\mathrm{Hom}_R(\tau X, \tau X) \cong \mathrm{Hom}_R(\tau^-\tau X, X) \cong \mathrm{Hom}_R(X, X)$$

and similarly

$$\mathrm{Ext}^1_R(\tau X, \tau X) \cong D\,\mathrm{Hom}_R(\tau^-\tau X, \tau X) \cong D\,\mathrm{Hom}_R(X, \tau X) \cong \mathrm{Ext}^1_R(X, X). \quad \square$$

Lemma 7.13. *The functor τ^- is right exact; that is, if $X \to Y \to Z \to 0$ is exact, then so too is $\tau^- X \to \tau^- Y \to \tau^- Z \to 0$.*

Dually, the functor τ is left exact.

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be exact. Then, given any finite dimensional module W, the functor $\mathrm{Hom}_R(-, \tau W)$ is left exact by Lemma 4.4, so we have the exact sequence of vector spaces

$$0 \to \mathrm{Hom}_R(Z, \tau W) \xrightarrow{g^*} \mathrm{Hom}_R(Y, \tau W) \xrightarrow{f^*} \mathrm{Hom}_R(X, \tau W).$$

We now use the Auslander-Reiten Formula to deduce that the following sequence is also exact

$$0 \to \mathrm{Hom}_R(\tau^- Z, W) \xrightarrow{(\tau^- g)^*} \mathrm{Hom}_R(\tau^- Y, W) \xrightarrow{(\tau^- f)} \mathrm{Hom}_R(\tau^- X, W).$$

This holds for all finite dimensional modules W, so using Lemma 4.4 again we conclude that

$$\tau^- X \xrightarrow{\tau^- f} \tau^- Y \xrightarrow{\tau^- g} \tau^- Z \to 0$$

is exact. (Again note that the lemma was stated for all modules W, but as in the proof of 3 implies 2 we just need to check against the three modules Z, $\mathrm{Coker}(g)$ and $\mathrm{Coker}(f)$, which are all finite dimensional.) □

7.5 Preprojective, regular and postinjective modules

Let X be any module. We say that X is preprojective provided $\tau^n X = 0$ for some $n > 0$, and is postinjective provided $\tau^{-n} X = 0$ for some $n > 0$. We say that X is regular provided it has neither preprojective nor postinjective summands.

Lemma 7.14. *Let X be an R-module.*

1. *X has no preprojective summands if and only if $\tau^{-n}\tau^n X \cong X$ for all $n > 0$.*
2. *X has no preinjective summands if and only if $\tau^n\tau^{-n} X \cong X$ for all $n > 0$.*
3. *X is regular if and only if $\tau^n\tau^{-n} X \cong X$ for all $n \in \mathbb{Z}$.*

Proof. 1. Assume first that X has no preprojective summands. Then it has no projective summands, so $\tau^-\tau X \cong X$. By induction we may assume that $\tau^{-n}\tau^n X \cong X$. If P is a projective summand of $\tau^n X$, then the preprojective $\tau^{-n} P$ is a summand of $\tau^{-n}\tau^n X \cong X$, a contradiction. Thus $\tau^n X$ has no projective summands, so

$$\tau^{-(n+1)}\tau^{n+1} X \cong \tau^{-n}\big(\tau^-\tau(\tau^n X)\big) \cong \tau^{-n}\tau^n X \cong X.$$

In general, we can write $X \cong P \oplus X'$ with P preprojective and X' having no preprojective summand. Then for $n \gg 0$ we have $\tau^n P = 0$, so $\tau^{-n}\tau^n X \cong \tau^{-n}\tau^n X' \cong X'$, so $\tau^{-n}\tau^n X \cong X$ if and only if $P = 0$.

2. This is dual to 1.

3. This follows from 1 and 2. □

Note. It is tempting to think that $\tau^m\tau^n \cong \tau^{m+n}$ for all $m, n \in \mathbb{Z}$, but this is not the case in general. However, it does hold on the subcategory of regular modules.

We remark that every module can be written essentially unique as $P \oplus X$, where P is preprojective and X has no preprojective summands, and also as $Y \oplus I$ where I is postinjective and Y has no postinjective summands. We then have the following vanishing results with respect to such decompositions.

Proposition 7.15. *Let P be preprojective, and X a module having no preprojective summands. Then*

$$\operatorname{Hom}_R(X,P)=0=\operatorname{Ext}^1_R(P,X).$$

Dually, let I be preinjective, and X a module having no preinjective summands. Then

$$\operatorname{Hom}_R(I,X)=0=\operatorname{Ext}^1_R(X,I).$$

Proof. We use the Auslander-Reiten Formula. By assumption we know that $\tau^n P=0$ for some $n>0$, whereas $\tau^{-m}\tau^m X\cong X$ for all $m>0$. Thus

$$\operatorname{Hom}_R(X,P)\cong\operatorname{Hom}_R(\tau^{-n}\tau^n X,P)\cong\operatorname{Hom}_R(\tau^n X,\tau^n P)=0.$$

Similarly, as τP is again preprojective, we have

$$\operatorname{Ext}^1_R(P,X)\cong D\operatorname{Hom}_R(X,\tau P)=0. \qquad \square$$

Corollary 7.16. *The class of preprojective modules is closed under taking extensions and submodules. Dually, the class of postinjective modules is closed under taking extensions and quotients. Finally, the class of regular modules is closed under taking extensions and images.*

Proof. Consider an exact sequence $0\to X\to Y\to Z$ with Z preprojective, say $\tau^m Z=0$ for some $m>0$. Using that τ is left exact, we have an exact sequence $0\to\tau^n X\to\tau^n Y\to\tau^n Z$ for all $n>0$, so $\tau^n X\cong\tau^n Y$ for all $n\geq m$. It follows that X is preprojective if and only if Y is preprojective, and hence that the class of preprojective modules is closed under taking extensions (take $Y\twoheadrightarrow Z$) and submodules (take $X=0$).

The result for postinjective modules is dual.

Finally, suppose $0\to X\to Y\to Z\to 0$ is exact, where X and Z are both regular. Applying $\operatorname{Hom}_R(-,P)$ shows that $\operatorname{Hom}_R(Y,P)=0$ for all preprojective modules P, and dually $\operatorname{Hom}_R(I,Y)=0$ for all postinjective modules I. Thus Y cannot have any preprojective or postinjective direct summand, so must be regular. Now if $f\colon X\to Z$ is any homomorphism, we have by Proposition 6.1 (6) a short exact sequence $0\to X\to Y\to Z\to 0$ such that $\operatorname{Im}(f)$ a direct summand of Y. Hence $\operatorname{Im}(f)$ is also regular. $\square$

We say that an module E is exceptional provided $\operatorname{End}_R(E)$ is a division algebra and $\operatorname{Ext}^1_R(E,E)=0$.

Lemma 7.17. *All indecomposable preprojective modules and all indecomposable postinjective modules are exceptional.*

Proof. All indecomposable projective and injective modules are exceptional. The result now follows from Corollary 7.12. $\square$

7.6 The Coxeter transformation

Recall that the Grothendieck group Γ has basis $e_i:=[S_i]$ and comes equipped with the non-degenerate bilinear form $\langle -,-\rangle$. Also, we have the indecomposable projective modules $P_i:=\varepsilon_i R$, and the injective modules $I_i:=D(R\varepsilon_i)$.

We introduce a partial order on Γ by saying that $x \geq 0$ provided $x = \sum_i x_i e_i$ with $x_i \geq 0$ for all i. We write Γ_+ for the set of positive elements. We observe that every $x \in \Gamma$ can be written as $x = x_+ - x_-$ with $x_\pm \in \Gamma_+$ (and with disjoint support). Also, Γ_+ coincides with the classes of R-modules $[X]$. (To see that this is onto, we can take the classes of semisimple modules.)

Lemma 7.18. *The classes of the indecomposable projectives $p_i := [P_i]$ form a basis for Γ, and $\langle p_i . e_j \rangle = \delta_{ij} \dim_k A_i$.*

Dually, the injective modules $I_i := D(R\varepsilon_i)$ are indecomposable, their classes $q_i := [I_i]$ for a basis for Γ, and $\langle e_i, q_j \rangle = \delta_{ij} \dim_k A_i$.

Proof. We have the short exact sequence $0 \to \varepsilon_i J \to P_i \to S_i \to 0$, where $\varepsilon_i J$ is again finite dimensional projective. Thus $e_i = [S_i]$ equals $[P_i] - [\varepsilon_i J]$, so lies in the span of the p_i. Next,

$$\langle [P_i], [S_j] \rangle = \dim_k \operatorname{Hom}_R(P_i, S_j) = \dim_k A_j \varepsilon_i = \delta_{ij} \dim_k A_i.$$

Finally we note that if $x = \sum_i x_i p_i$, then $\langle x, e_i \rangle = x_i d_i$, so the p_i must be linearly independent.

Similarly, dualising the analogous sequence for left R-modules yields the short exact sequence $0 \to S_i \to I_i \to D(J\varepsilon_i) \to 0$, where $D(J\varepsilon_i)$ is again a finite direct sum of the I_j. Thus the q_i span Γ. Also

$$\langle [S_j], [I_i] \rangle = \dim_k \operatorname{Hom}_R(S_j, I_i) = \dim_k D(S_j \otimes_R R\varepsilon_i) = \dim_k A_j \varepsilon_i = \delta_{ij} \dim_k A_i.$$

Finally, if $y = \sum_i y_i q_i$, then $\langle e_i, y \rangle = d_i y_i$, so the q_i are linearly independent.

To see that the I_i are indecomposable (which we should probably have done earlier), we note that

$$\operatorname{End}_R(I_i) \cong \operatorname{End}_R(R\varepsilon_i) \cong \varepsilon_i R \varepsilon_i \cong A_i,$$

which is a division ring. □

Proposition 7.19. *There exists a unique automorphism c of Γ, called the Coxeter transformation, such that*

$$\langle y, c(x) \rangle = -\langle x, y \rangle \quad \text{for all } x, y \in \Gamma.$$

Proof. Using that p_i and q_i form two bases for Γ, we can define an automorphism c of Γ via $c(p_i) = -q_i$. From the previous lemma we have $\langle p_i, e_j \rangle = \delta_{ij} = \langle e_j, q_i \rangle$ for all i, j, so by bilinearity we obtain $\langle x, y \rangle = -\langle y, c(x) \rangle$ for all $x, y \in \Gamma$. The uniqueness is clear, since the form is non-degenerate. □

Corollary 7.20. *We have $\langle c(x), c(y) \rangle = \langle x, y \rangle$ for all $x, y \in \Gamma$.*

Proof. We have $\langle x, y \rangle = -\langle y, c(x) \rangle = \langle c(x), c(y) \rangle$. □

An important consequence is that we can relate the the Coxeter transformation to the Auslander-Reiten translate.

Proposition 7.21. *Suppose X has no projective summand. Then $[\tau X] = c[X]$.*

Dually, if Y no injective summand, then $[\tau^- Y] = c^-[Y]$.

Proof. By assumption we have $X \cong \tau^{-}\tau X$, so together with the Auslander-Reiten Formula we compute that

$$\begin{aligned}\langle [X],[Y]\rangle &= \dim_k \operatorname{Hom}_R(\tau^{-}\tau X, Y) - \dim_k \operatorname{Ext}^1_R(X,Y)\\ &= \dim_k \operatorname{Ext}^1_R(Y,\tau X) - \dim_k \operatorname{Hom}_R(Y,\tau X) = -\langle [Y],[\tau X]\rangle.\end{aligned}$$

Thus $\langle y,[\tau X]\rangle = -\langle [X],y\rangle = \langle y, c[X]\rangle$ for all $y \in \Gamma_+$, and hence for all $y \in \Gamma$ by linearity. Since the form is non-degenerate, we conclude that $[\tau X] = c[X]$.

The second result is dual. □

We finish with the following results, classifying the indecomposable preprojective and postinjective modules in terms of their images in the Grothendieck group.

Proposition 7.22. *Let X be indecomposable. Then X is preprojective if and only if there exists $r > 0$ with $c^r[X] > 0 > c^{r+1}[X]$, in which case, taking the minimal such r, we have $X \cong \tau^{-r}P_i$, where i is uniquely determined by $\langle c^r[X], e_i\rangle \neq 0$.*

Dually, an indecomposable Y is postinjective if and only if there exists $r > 0$ with $c^{-r}[Y] > 0 > c^{-r-1}[Y]$, in which case, taking the minimal such r, we have $Y \cong \tau^r I_i$, where i is uniquely determined by $\langle e_i, c^{-r}[Y]\rangle \neq 0$.

Proof. Let X be indecomposable. Then X is projective if and only if $c[X] < 0$, in which case $X \cong P_i$ where $\langle [X], e_i\rangle \neq 0$. If X is not projective, then τX is indecomposable, $X \cong \tau^{-}\tau X$, and $c[X] = [\tau X] > 0$. Iterating this we see that $X \cong \tau^{-r}P_i$ for some i if and only if $[X], c[X], \dots, c^r[X]$ are all positive, $c^{r+1}[X] < 0$, and $\langle c^r[X], e_i\rangle \neq 0$. □

Theorem 7.23. *The map $X \mapsto [X]$ induces a bijection between isomorphism classes of indecomposable preprojectives and the set $\{c^{-r}(p_i),\ r \geq 0\} \cap \Gamma_+$.*

Dually, it induces a bijection between the isomorphism classes of indecomposable injectives and the set $\{c^r(q_i),\ r \geq 0\} \cap \Gamma_+$.

Proof. If $X \cong \tau^{-r}P_i$ is indecomposable preprojective, then $[X] = c^{-r}(p_i) > 0$. Moreover, by the previous proposition, if Y is any indecomposable with $[Y] = [X]$, then $Y \cong X$, and so the map is injective on isomorphism classes of indecomposable preprojectives.

It remains to prove surjectivity. It follows from the above remarks that the image consists of those elements $x > 0$ such that there exists $r > 0$ with $x, c(x), \dots, c^r(x) > 0$ and $c^r(x) = p_i$ for some i. There is thus one subtlety: we may have $c^r(x) = p_i$ for some $r > 0$ and some i, but $c^s(x) \not> 0$ for some $0 < s < r$.

Suppose therefore that $a, b > 0$ are minimal such that $c^{1-a}(p_i) > 0 \not< c^{-a}(p_i)$ and $c^{1-a-b}(p_i) \not> 0 < c^{-a-b}(p_i)$. Then $\tau^{1-a}P_i$ must be an indecomposable injective, say isomorphic to I_j, and so $c^{-a}(p_i) = -p_j$. Similarly, $\tau^{1-b}P_j$ must be an indecomposable injective, say isomorphic to I_m, so $c^{-a-b}(p_i) = p_m$.

Repeating in this way, we see that given $x > 0$ with $c^r(x) = p_i$ for some $r > 0$ and some i, there exists $s > 0$ and j such that $x, c(x), \dots, c^s(x) = p_j$ are all positive, and hence that $x = [\tau^{-s}P_j]$. □

7.7 Examples

Let $R = A \oplus J$ be a finite dimensional hereditary k-algebra. We have seen that the bilinear form $\langle -,- \rangle$ on the Grothendieck group Γ depends only on the dimensions $d_i := \dim_k A_i$ and $m_{ij} := \varepsilon_i(J/J^2)\varepsilon_j$. Moreover, the classes $p_i := [P_i]$ and $q_i := [I_i]$ are completely determined by the identities

$$\langle p_i, e_j \rangle = \delta_{ij} d_i = \langle e_i, q_j \rangle,$$

and the Coxeter transformation c is given by $c(p_i) = -q_i$. Finally, the classes of the indecomposable preprojectives are computed as $p_i, c^{-1}(p_i), c^{-2}(p_i), \ldots$ stopping only if we reach some q_j, and dually for the classes of the indecomposable postinjectives. Thus all this information can be calculated without explicitly describing the indecomposable modules themselves.

For example, suppose

$$R = \begin{pmatrix} K & 0 \\ K & k \end{pmatrix}$$

for some field extension K/k of degree n. Then the bilinear form $\langle -,- \rangle$ on $\Gamma = \mathbb{Z}^2$ is represented (with respect to the basis $e_i = [S_i]$) by the matrix

$$\langle -,- \rangle \leftrightarrow \begin{pmatrix} n & 0 \\ -n & 1 \end{pmatrix},$$

the classes of the indecomposable projectives and injectives are

$$p_1 = (1,0), \quad p_2 = (1,1) \quad \text{and} \quad q_1 = (1,n), \quad q_2 = (0,1),$$

the Coxeter transformation c acts via the matrix

$$c \leftrightarrow \begin{pmatrix} -1 & -n \\ 1 & n-1 \end{pmatrix} \quad \text{and} \quad c^{-1} \leftrightarrow \begin{pmatrix} n-1 & n \\ -1 & -1 \end{pmatrix}.$$

We can therefore compute the classes in Γ of the indecomposable preprojective and postinjective modules. We display these as

$$\begin{array}{ccccccccccccccc} & & p_2 & & & & c^{-1}(p_2) & & & & & & c^2(q_2) & & & & c(q_2) & & & & q_2 \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \cdots & & \cdots & & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ p_1 & & & & c^{-1}(p_1) & & & & c^{-2}(p_1) & & & & & & c(q_1) & & & & q_1 & & \end{array}$$

Case $n = 1$.

$$\begin{array}{ccccc} & & (1,1) & & \\ & \nearrow & & \searrow & \\ (1,0) & & & & (0,1) \end{array}$$

Case $n = 2$.

$$\begin{array}{ccccccc} & & (1,1) & & & & (0,1) \\ & \nearrow & & \searrow & & \nearrow & \\ (1,0) & & & & (1,2) & & \end{array}$$

Case $n = 3$.

$$
\begin{array}{ccccccccc}
 & & (1,1) & & & & (1,2) & & & & (0,1) \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
(1,0) & & & & (2,3) & & & & (1,3) & &
\end{array}
$$

Case $n = 4$.

$$
\begin{array}{ccccccccccccccccc}
 & & (1,1) & & & & (2,3) & & & & & & (2,5) & & & & (1,3) & & & & (0,1) \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \cdots & \cdots & & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
(1,0) & & & & (3,4) & & & & (5,8) & & & & & & (3,8) & & & & (1,4) & &
\end{array}
$$

Thus when $n = 1, 2, 3$ we see that there are only finitely many indecomposable preprojective and postinjective modules, and these classes coincide. When $n = 4$, however, there are infinitely many preprojective indecomposables and infinitely many postinjective modules, and these classes are distinct.

7.8 Gabriel's Theorem

We say that R has finite representation type provided there are only finite many indecomposable modules up to isomoprhism. We say that the bilinear form on the Grothendieck group is positive definite provided $\langle x, x\rangle > 0$ for all $x \neq 0$.

Theorem 7.24 (Gabriel)**.** *The following are equivalent for a finite dimensional hereditary algebra R.*

1. *R has finite representation type.*
2. *The classes of preprojective and postinjective modules coincide.*
3. *There are no regular modules.*
4. *All indecomposables are exceptional.*
5. *The bilinear form is positive definite.*

Proof. $1 \Rightarrow 2$. We have seen that the map $X \mapsto [X]$ is injective on isomorphism classes of preprojective modules. Thus, since R has finite representation type, we must have that $\tau^{-r}P_i = 0$ for $r \gg 0$, and hence that each preprojective is necessarily postinjective. Dually, every postinjective is necessarily preprojective, so the result follows.

$2 \Rightarrow 1{,}3$. Every module has a projective resolution, and since every projective is postinjective, it follows that every module is postinjective. Dually, every module is preprojective, so there are no regular modules. It now follows that every indecomposable is of the form $\tau^{-r}P_i$ for some $r > 0$ and some i, and also that $\tau^{-s}P_i = 0$ for $s \gg 0$. Hence R has finite representation type.

$3 \Rightarrow 4$. We know that every indecomposable preprojective and postinjective is exceptional, and by assumption there are no indecomposable regular modules.

$4 \Rightarrow 5$. Take $x > 0$. Then we can write $x = [X]$ for some R-module X, and we take such an X with $\dim_k \mathrm{End}_R(X)$ minimal. If X is indecomposable, then it is exceptional, and so $\langle x, x\rangle = \dim \mathrm{End}_R(X) > 0$. Assume therefore that $X = X' \oplus X''$ is decomposable. If $\mathrm{Ext}^1_R(X'', X') \neq 0$, then we have a non-split short exact sequence $0 \to X' \to Y \to X'' \to 0$. Applying $\mathrm{Hom}_R(-,Y)$ and then $\mathrm{Hom}_R(X,-)$ yields that

$$\dim_k \mathrm{End}_R(Y) \leq \dim_k \mathrm{Hom}_R(X,Y) \leq \dim_k \mathrm{End}_R(X).$$

Moreover, the second inequality is strict, since otherwise we could lift the projection $X \to X''$ to a map $X \to Y$, yielding a section $X'' \to Y$ and showing that the sequence is split. Since $[X] = [Y]$, this contradicts the fact that $\dim_k \mathrm{End}_R(X)$ is minimal. Similarly $\mathrm{Ext}^1_R(X', X'') = 0$, and so

$$(x', x'') = \dim_k \mathrm{Hom}_R(X', X'') + \dim_k \mathrm{Hom}_R(X'', X') \geq 0,$$

where $x' := [X']$ and $x'' := [X'']$. By induction we know that $\langle x', x' \rangle > 0$ and $\langle x'', x'' \rangle > 0$, so

$$\langle x, x \rangle = \langle x' + x'', x' + x'' \rangle = \langle x', x' \rangle + \langle x'', x'' \rangle + (x', x'') > 0.$$

In general we can write $x = x_+ - x_-$ with $x_\pm \geq 0$ and having disjoint support. Then $(x_+, x_-) \leq 0$, so

$$\langle x, x \rangle = \langle x_+ - x_-, x_+ - x_- \rangle = \langle x_+, x_+ \rangle + \langle x_-, x_- \rangle - (x_+, x_-) \geq 0$$

with equality if and only if $x_+ = x_- = 0$.

$5 \Rightarrow 2$. We begin by observing that if $X \cong \tau^- r P_i$ is indecomposable preprojective, then $[X] = c^r(p_i)$ and $\langle [X], [X] \rangle = \langle p_i, p_i \rangle = d_i$, where $d_i = \dim_k A_i$.

Now, by extending scalars, we have a positive definite, symmetric bilinear form $(-,-)$ on $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma \cong \mathbb{Q}^n$. As in the Gram-Schmidt orthonormalistaion algorithm, we can construct an orthogonal basis in $\mathbb{Q}^n$. It follows that the ball $\{x \in \mathbb{Q}^n : (x,x) \leq d\}$ is bounded, so contains only finitely many lattice points, that is points in Γ. In particular, taking d to be the maximum of the d_i, we see that there are only finitely many $x \in \Gamma_+$ such that $(x,x) \leq d$, and hence only finitely many indecomposable preprojective and postinjective modules.

Since the map $X \mapsto [X]$ is injective on isomorphism classes of preprojective modules, we must have $\tau^{-r} P_i = 0$ for $r \gg 0$, and hence all preprojective modules are necessarily postinjective. Dually every postinjective module is necessarily preprojective, so the two classes coincide. □

We finish by refining this result to show that if R is an indecomposable algebra, then it is representation finite if and only if the classes of preprojectives and postinjectives intersect non-trivially.

Lemma 7.25. *The following are equivalent for a finite dimensional algebra R.*

1. *R is not an indecomposable algebra.*
2. *We can write $R = P' \oplus P''$ with $\mathrm{Hom}_R(P', P'') = 0 = \mathrm{Hom}_R(P'', P')$.*
3. *We can write $R/J = S' \oplus S''$ with $\mathrm{Hom}_R(S', S'') = 0 = \mathrm{Hom}_R(S'', S')$ and $\mathrm{Ext}^1_R(S', S'') = 0 = \mathrm{Ext}^1_R(S'', S')$.*

Proof. $2 \Rightarrow 1$. Suppose we have such a direct sum $R = P' \oplus P''$. Then

$$R \cong \mathrm{End}_R(R) \cong \mathrm{End}_R(P' \oplus P'') \cong R' \times R''.$$

1 ⇒ 3. Suppose $R \cong R' \times R''$. Then $\varepsilon' = (1,0)$ and $\varepsilon'' = (0,1)$ are orthogonal central idempotents. If X' is an R'-module, and X'' an R''-module, then $X' = X'\varepsilon'$ and $X'' = X''\varepsilon''$, so $\mathrm{Hom}_R(X', X'') = 0 = \mathrm{Hom}_R(X'', X')$. In particular, take $S' := R'/J'$ and $S'' := R''/J''$, so that $R/J \cong S' \oplus S''$. Then $\mathrm{Hom}_R(S', S'') = 0$, and also $\mathrm{Ext}^1_R(S', S'') \cong \mathrm{Hom}_R(J', S'') = 0$. Similarly $\mathrm{Hom}_R(S'', S') = 0 = \mathrm{Ext}^1_R(S'', S')$.

3 ⇒ 2. Suppose we have such a decomposition $R/J = S' \oplus S''$. Then we can decompose $R = P' \oplus P''$ such that $P'/P'J \cong S'$ and $P''/P''J \cong S''$. (To see this, first decompose $R = \bigoplus_i P_i$ as a direct sum of indecomposable projectives. Then $R/J \cong \bigoplus_i (P_i/P_iJ)$ is a direct sum of simples; now apply the Krull-Remak-Schmidt Theorem.)

We know that J is nilpotent, say $J^{n+1} = 0$. Consider the short exact sequences $0 \to P'J^{r+1} \to P'J^r \to Q'_r \to 0$, so each Q'_r is semisimple module, $Q'_0 = S'$, and $Q'_n = P'J^n$. Next, applying $\mathrm{Hom}_R(-, S'')$ yields

$$\mathrm{Hom}_R(P'J^{r+1}, S'') \rightarrowtail \mathrm{Ext}^1_R(Q'_r, S'')$$

and

$$\mathrm{Hom}_R(Q'_r, S'') \xrightarrow{\sim} \mathrm{Hom}_R(P'J^r, S'').$$

Thus $\mathrm{Hom}_R(P', S'') \cong \mathrm{Hom}_R(S', S'') = 0$, and

$$\begin{aligned} \mathrm{Hom}_R(Q'_r, S'') = 0 \quad &\Longrightarrow \quad \mathrm{Ext}^1_R(Q'_r, S'') = 0 \\ &\Longrightarrow \quad \mathrm{Hom}_R(P'J^{r+1}, S'') = 0 \quad \Longrightarrow \quad \mathrm{Hom}_R(Q'_{r+1}, S'') = 0. \end{aligned}$$

The first implication follows since every simple module is a summand of $R/J = S' \oplus S''$, so if Q is semisimple and $\mathrm{Hom}_R(Q, S'') = 0$, then Q is a summand of $(S')^m$ for some m, and hence $\mathrm{Ext}^1_R(Q, S'') = 0$.

Now we similarly have $\mathrm{Hom}_R(P'', S') = 0$, so $\mathrm{Hom}_R(P'', Q'_r) = 0$ for all r. Thus applying $\mathrm{Hom}_R(P'', -)$ yields

$$\mathrm{Hom}_R(P'', P'J^r) \cong \mathrm{Hom}_R(P'', P'J^{r+1}) \quad \text{for all } r.$$

Hence $\mathrm{Hom}_R(P'', P') \cong \mathrm{Hom}_R(P'', P'J^{n+1}) = 0$. Analogously $\mathrm{Hom}_R(P', P'') = 0$. □

Theorem 7.26. *Let R be an indecomposable, finite dimensional hereditary algebra. Then R has finite representation type if and only if some non-zero module is both preprojective and postinjective.*

Proof. We have already seen that R has finite representation type if and only if every projective module is postinjective. Suppose now that X is non-zero and both preprojective and postinjective. Then the same holds for every indecomposable summand of X, and applying τ, we see that some (indecomposable) projective module is postinjective. So we need to prove that, under the assumption that R is connected, if one projective module is postinjective, then every projective module is postinjective.

Recall that $m_{ij} = \dim_k \varepsilon_i(J/J^2)\varepsilon_j = \dim_k \mathrm{Ext}^1_R(S_i, S_j)$. Suppose $m_{ij} \neq 0$. Then we know that there is a non-zero map $P_j \to P_i$, so if P_j is postinjective, then so too is P_i. On the other hand, we claim that there is also a non-zero map $P_i \to \tau^- P_j$, so if P_i is postinjective, then so too is $\tau^- P_j$, and hence also P_j. It follows that the subset of vertices i for which P_i is postinjective is either empty or everything, and in the latter case we know that R has finite representation type.

It remains to prove the claim. We begin by observing that the map $P_j \to P_i$ is a monomorphism by Lemma 6.16, so P_j is not injective. Thus

$$\begin{aligned}\dim_k \operatorname{Hom}_R(P_i, \tau^- P_j) &= \langle p_i, c^{-1}(p_j)\rangle = \langle c(p_i), p_j\rangle \\ &= -\langle q_i, p_j\rangle = \dim_k \operatorname{Ext}^1_R(I_i, P_j).\end{aligned}$$

Next, applying $\operatorname{Hom}_R(-, P_j)$ to the monomorphism $S_i \rightarrowtail I_i$, and $\operatorname{Hom}_R(S_i, -)$ to the pimorphism $P_j \twoheadrightarrow S_j$, so yields epimorphisms

$$\operatorname{Ext}^1_R(I_i, P_j) \twoheadrightarrow \operatorname{Ext}^1_R(S_i, P_j) \twoheadrightarrow \operatorname{Ext}^1_R(S_i, S_j).$$

So $m_{ij} \neq 0$ implies $\operatorname{Ext}^1_R(I_i, P_j) \neq 0$, and hence also $\operatorname{Hom}_R(P_i, \tau^- P_j) \neq 0$. □

Remark. We saw in the proof that

$$\dim_k \operatorname{Hom}_R(P_i, \tau^- P_j) = \dim_k \operatorname{Ext}^1_R(I_i, P_j).$$

In fact, we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(P_i, \tau^- P_j) &\cong (\tau^- P_j)\varepsilon_i \cong \operatorname{Ext}^1_R(DR, P_j)\varepsilon_i \\ &\cong \operatorname{Ext}^1_R(\varepsilon_i(DR), P_j) \cong \operatorname{Ext}^1_R(D(R\varepsilon_i), P_j) \cong \operatorname{Ext}^1_R(I_i, P_j).\end{aligned}$$

8 Cartan data and root systems

In this section we investigate the possible bilinear forms which can arise as the Euler form of a finite dimensional hereditary algebra, and introduce some of the related combinatorial structures such as the root system and the Weyl group.

8.1 Cartan data

A symmetrisable generalised Cartan matrix is an integer matric $C \in \mathbb{M}_n(\mathbb{Z})$ such that $c_{ii} = 2$ for all i, $c_{ij} \leq 0$ for all $i \neq j$, and there exist positive integers d_i such that $d_i c_{ij} = d_j c_{ji}$ for all i, j. Setting $D := \mathrm{diag}(d_1, \dots, d_n)$ we see that $B := DC$ is a symmetric matrix, called a Cartan datum. Note that we can recover D, and hence C, from B, since $b_{ii} = 2d_i$.

Every Cartan datum B yields a symmetric bilinear form $(-,-)$ on the lattice $\Gamma := \mathbb{Z}^n$, called the root lattice. The standard basis elements e_i are called simple roots, and we equip Γ with a partial order by declaring $x = \sum_i x_i e_i \geq 0$ provided $x_i \geq 0$ for all i. We also define the support of an element $x = \sum_i x_i e_i$ to be those i for which $x_i \neq 0$.

We can represent a symmetrisable generalised Cartan matrix C by its Dynkin diagram Δ, which is the valued graph having vertices $1, \dots, n$ and a valued edge $i \xrightarrow{(|c_{ij}|,|c_{ji}|)} j$ whenever $c_{ij} \neq 0$. We usually omit the label $(1,1)$ for simplicity; it is also common to relace a label (m, m) by a single m, or by m egdes.

We say that B is connected provided its Dynkin diagram Δ is connected. We say that B, or Δ, is finite if the corresponding bilinear form is positive definite; it is affine if the form is positive semidefinite but not positive definite; and it is wild if the form is indefinite.

8.2 The Weyl group

Let B be a Cartan datum. We define the Weyl group W to be the group having generators s_i for $1 \leq i \leq n$ subject to the relations $s_i^2 = 1$, and $(s_i s_j)^{m_{ij}} = 1$ for $i \neq j$, where the exponents m_{ij} are determined according to the table

$c_{ij}c_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

The Weyl group acts naturally on the root lattice Γ as follows.

Lemma 8.1. *There is a representation $\rho\colon W \to \mathrm{Aut}(\Gamma)$ sending s_i to the reflection*

$$\rho(s_i)\colon x \mapsto x - \tfrac{1}{d_i}(x, e_i)e_i.$$

In particular, $(w(x), w(y)) = (x, y)$ for all $x, y \in \Gamma$ and all $w \in W$.

Proof. Define r_i via the formula $x \mapsto x - \frac{1}{d_i}(x, e_i)e_i$. Then it is easy to check that $r_i^2 = 1$ and that $(r_i(x), y) = (x, r_i(y))$. It remains to compute the order of $r_i r_j$ for $i \neq j$.

By reordering the basis elements, it is enough to compute the order of $r_1 r_2$. Now the matrix representing this automorphism is $\left(\begin{smallmatrix} M & 0 \\ N & I \end{smallmatrix}\right)$, where I is the identity matrix and M describes the action of r_1r_2 on $\mathbb{Z}e_1+\mathbb{Z}e_2$. Now $r_i(e_j)=e_j-c_{ij}e_i$, so we can compute M explicitly to be

$$M=\begin{pmatrix} c_{12}c_{21}-1 & -c_{21} \\ c_{12} & -1 \end{pmatrix}.$$

Thus $(r_1r_2)^a$ is represented by the matrix $\left(\begin{smallmatrix} M^a & 0 \\ Np_1(M) & I \end{smallmatrix}\right)$, where $p_a(t)=(t^a-1)/(t-1)=t^{a-1}+\cdots+t+1$.

Now, the characteristic polynomial of M is $t^2+(2-c_{12}c_{21})t+1$, so if $c_{12}c_{21}=1,2,3$, then the eigenvalues of M are complex conjugate primitive m-th roots of unity, where $m=m_{12}$ is given by the table above. In particular, $p_m(M)=0$ and $M^m=1$, so $(r_1r_2)^m=1$. If instead $c_{12}c_{21}=0$, then necessarily $c_{12}=0=c_{21}$, so $M=-1$ and $(r_1r_2)^2=I$. Finally, if $c_{12}c_{21}\geq 4$, then M has a real eigenvalue larger than 1, so M, and hence also r_1r_2, has infinite order. □

We call the generators s_i of W the simple reflections. A Coxeter element in W is then any element of the form $c=s_{i_1}\cdots s_{i_n}$, where each simple reflection occurs precisely once.

8.3 Generalised Cartan lattices

Let B be a Cartan datum, Γ the root lattice, and W the Weyl group. Given an ordering of the simple roots $i_1<\cdots<i_n$, we have the corresponding Coxeter element $c=s_{i_n}\cdots s_{i_1}$, and can construct a non-symmetric bilinear form $\langle -,-\rangle$ such that $\langle e_i,e_j\rangle=0$ whenever $i<j$, and $\langle x,y\rangle+\langle y,x\rangle=(x,y)$ for all $x,y\in\Gamma$. We call Γ, together with this bilinear form $\langle -,-\rangle$, a generalised Cartan lattice.

Proposition 8.2. *The Grothendieck group of a finite dimensional hereditary algebra is always a generalised Cartan lattice, and every such lattice arises in this way.*

Proof. Let R be a finite dimensional hereditary k-algebra. We know that R is a triangular algebra, so $\varepsilon_iJ\varepsilon_j=0$ for all $i<j$, and hence the matrix representing $\langle -,-\rangle$ is lower triangular. We set $d_i:=\langle e_i,e_i\rangle=\dim_k A_i>0$. For $i\neq j$ we have $\langle e_i,e_j\rangle=-\dim_k \mathrm{Ext}^1_R(S_i,S_j)\leq 0$. Moreover, this is a right module for $\mathrm{End}_R(S_i)=A_i$, and a left module for $\mathrm{End}_R(S_j)=A_j$. Since each A_i is a division algebra, we know that $\dim_k \mathrm{Ext}^1_R(S_i,S_j)$ is divisible by both d_i and d_j. It follows that if B is the matrix representing the symmetric bilinear form $(-,-)$, and $D=\mathrm{diag}(d_1,\dots,d_n)$, then $D^{-1}B$ is a symmetrisable generalised Cartan matrix.

Conversely, let Γ be any generalised Cartan lattice such that $\langle e_i,e_j\rangle=0$ for $i<j$. Fix a finite field k, and an algebraic closure $\bar{k}$. Then for each $d\geq 0$, there is a unique subfield k_d of $\bar{k}$ containing k, and $k_d\subset k_e$ if and only if d divides e. We now set $A:=\prod_i A_i$, where A_i/k is the field extension of degree $d_i:=\langle e_i,e_i\rangle$ inside $\bar{k}$. Similarly, for $i>j$ such that $m_{ij}=-\langle e_i,e_j\rangle\neq 0$, take M_{ij}/k to be the field extension of degree m_{ij} inside $\bar{k}$, and set $M:=\bigoplus_{i,j}M_{ij}$. Then A_i and A_j are both subfields of M_{ij}, so M_{ij} is naturally an A_i-A_j-bimodule on which k acts centrally. Hence M is naturally an A-bimodule on which k acts centrally, and $M^{\otimes n}=0$, so $R:=T_A(M)$ is a finite dimensional hereditary k-algebra, whose Grothendieck group (with Euler form) is precisely Γ. □

Let Γ be a generalised Cartan lattice, with corresponding Coxeter element $c = s_n \cdots s_1$. Then $\langle -, - \rangle$ is lower triangular, and we define

$$p_1 := e_1 \quad \text{and} \quad p_i := s_1 s_2 \cdots s_{i-1}(e_i),$$

and similarly

$$q_n := e_n \quad \text{and} \quad q_i := s_n s_{n-1} \cdots s_{i+1}(e_i).$$

Lemma 8.3. *We have $\langle p_i, e_j \rangle = \delta_{ij} d_i = \langle e_j, q_i \rangle$. Moreover, $c(p_i) = -q_i$.*

Proof. We begin by observing that, as $s_i(e_i) = -e_i$, we have

$$c(p_i) = s_n \cdots s_1 s_1 \cdots s_{i-1}(e_i) = -s_n \cdots s_{i+1}(e_i) = -q_i.$$

Next, it is clear that $p_i - e_i$ lies in the span of $e_1, \dots, e_{i-1}$, so $\langle p_i, e_j \rangle = 0$ for all $i < j$, and also $\langle p_i, e_i \rangle = d_i$. Now suppose $i > j$, and set $a := s_{j+1} \cdots s_{i-1}(e_i)$. Then $\langle e_j, a \rangle = 0$, so $s_j(a) = a - \frac{1}{d_j} \langle a, e_j \rangle e_j$. Also, $p_i = s_1 \cdots s_j(a)$, so $p_i - s_j(a)$ lies in the span of $e_1, \dots, e_{j-1}$. Therefore

$$\langle p_i, e_j \rangle = \langle s_j(a), e_j \rangle = \langle a - \tfrac{1}{d_j} \langle a, e_j \rangle e_j, e_j \rangle = 0.$$

The proof that $\langle e_j, q_i \rangle = \delta_{ij} d_i$ is entirely analogous. $\square$

It follows from this that if R is any finite dimensional hereditary algebra whose Grothendieck group is given by Γ, then $p_i = [P_i]$ and $q_i = [I_i]$ give the classes of the indecomposable projective and injective modules, and the Coxeter element c acts as the Coxeter transformation for R defined earlier.

8.4 Real roots and reflections

Let B be a Cartan datum, and W be the associated Weyl group. Recall that the generators s_i of W are called the simple reflections. In general we define the set of all reflections to be

$$T := \{ w s_i w^{-1} : w \in W, 1 \leq i \leq n \}.$$

The set of real roots in Γ is

$$\Phi^{\mathrm{re}} := \{ w(e_i) : w \in W, \ 1 \leq i \leq n \}.$$

Note that if $t = w s_i w^{-1}$ is a reflection, and $a = w(e_i)$ the corresponding real root, then t acts on Γ by $t(x) = x - \frac{1}{d_i}(x, a)a$. Thus this action depends only on a, so we can write $\rho(t) = s_a$. Note also that $\frac{2(x,a)}{(a,a)} = \frac{1}{d_i}(w^{-1}(x), e_i) \in \mathbb{Z}$.

Given an element $w \in W$, we define its length $\ell(w)$ to be the minimal length of an expression $w = s_{i_1} \cdots s_{i_m}$ involving the simple reflections, and call any such expression of minimal length a reduced expression. Clearly $\ell(w) = 0$ if and only if $w = 1$, and $\ell(w) = 1$ if and only if w is a simple reflection. It is clear that

$$\ell(w'w) \leq \ell(w') + \ell(w) \quad \text{and similarly} \quad \ell(w') \leq \ell(w'w) + \ell(w^{-1}).$$

Since w and w^{-1} have the same length, we obtain

$$|\ell(w'w) - \ell(w')| \leq \ell(w).$$

On the other hand, we know that $\det(\rho(s_i)) = -1$ for each i, so $\det(\rho(w)) = (-1)^{\ell(w)}$. In particular, $\ell(ws_i) = \ell(w) \pm 1$, and every reflection $t \in T$ has odd length.

The next proposition is fundamental to the theory of Weyl groups, since it relates the structure of the group to the geometry of the lattice Γ.

Proposition 8.4. *We have $\ell(ws_i) > \ell(w)$ if and only if $w(e_i) > 0$, and similarly $\ell(ws_i) < \ell(w)$ if and only if $w(e_i) < 0$.*

Proof. It is enough to prove that $\ell(ws_i) < \ell(w)$ implies $w(e_i) < 0$. For, we know that $\ell(ws_i) = \ell(w) \pm 1$, and if $\ell(ws_i) > \ell(w)$, then $ws_i(e_i) < 0$, so $w(e_i) > 0$.

Consider first the rank two case. Thus we have the Cartan datum

$$B = \begin{pmatrix} 2d_1 & -b \\ -b & 2d_2 \end{pmatrix}, \quad b \geq 0, \text{ divisible by } d_1, d_2.$$

Note that W is a (possibly infinite) dihedral group, every element of W is an alternating product of s_1 and s_2, and the reflections are precisely the elements of odd length. We also observe that every real root is either positive or negative. For, consider $a = w(e_i) = a_1e_1 + a_2e_2$. Then $d_i = \frac{1}{2}(a,a) = d_1a_1^2 + d_2a_2^2 - ba_1a_2$, so if a_1 and a_2 have different signs, then the right hand side is strictly larger than d_1 and d_2, a contradiction.

Without loss of generality take w such that $\ell(ws_1) < \ell(w)$. Suppose first that w has odd length, so is a reflection. If $\ell(w) = 1$, then $w = s_1$ and $w(e_1) < 0$. Otherwise we can write $w = s_1s_as_1$ with s_a a reflection of length $\ell(s_a) = \ell(w) - 2$. Then $\ell(s_a) < \ell(s_as_1)$, so by induction $s_as_1(e_1) < 0$, whence $s_a(e_1) > 0$. We need to show that $s_1s_a(e_1) > 0$, so that $w(e_1) < 0$.

Assume therefore that $s_1s_a(e_1) < 0$. Then necessarily $s_a(e_1) = me_1$, and since $(s_a(e_1), s_a(e_1)) = (e_1, e_1)$, we must have $m = 1$. It follows that $(a, e_1) = 0$, so s_a and s_1 commute, implying $w = s_a$, a contradiction.

Now suppose that w has even length. Then we can write $w = s_as_1$ with $\ell(s_as_2) < \ell(s_a) < \ell(w)$. By induction we have $s_a(e_2) < 0$. We claim that $s_a(e_1) > 0$, so that $w(e_1) < 0$.

Assume therefore that $s_a(e_1)$ and $s_a(e_2)$ are both negative, and write $s_a = vs_iv^{-1}$, so $a = v(e_i)$. Then the matrix representing s_a has non-positive coefficients, zero trace (since s_i has zero trace), and squares to the identity, so it must be $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. Thus $s_a(e_1) = -e_2$, so

$$-2d_1 = (e_1, s_a(e_2)) = (s_a(e_1), e_2) = -2d_2,$$

and $d_1 = d_2$. Next $s_a(a) = -a$, so $a = m(e_1 + e_2)$, and since $a = v(e_i)$ we know that $(a, a) = (e_i, e_i) = 2d_1$. Thus

$$2d_1 = (a, a) = m^2(e_1 + e_2, e_1 + e_2) = 2m^2(2d_1 - b),$$

and since d_1 divides b we must have $m^2(2 - b/d_1) = 1$, so $m = \pm 1$ and $b = d_1$. It follows that $(s_1s_2)^3 = 1$, so the Weyl group is the dihedral group of order six, and $s_a = s_1s_2s_1 = s_2s_1s_2$. In particular, s_a has maximal length, contradicting the fact that $\ell(w) > \ell(s_a)$. Thus we must have $s_a(e_1) > 0$ as claimed.

We now prove the result in the general case. Again, we can take w such that $\ell(ws_1) < \ell(w)$. If $\ell(w) = 1$, then $w = s_1$ and $w(e_1) < 0$. Otherwise we may assume that $\ell(ws_1s_2) < \ell(ws_1)$. This shows that we can write $ws_1 = w'w''$ such that $\ell(w)-1=\ell(ws_1)=\ell(w')+\ell(w'')$ and w'' a product of s_1 and s_2. Now take such an expression with $\ell(w')$ minimal. Then $\ell(w')<\ell(w's_1),\ell(w's_2)<\ell(w)$, so by induction $w'(e_1),w'(e_2)>0$. On the other hand, we have $\ell(w''s_1)>\ell(w'')$, so by the rank two case we have $w''(e_1)=x_1e_1+x_2e_2>0$. Then $w(e_1)=-x_1w'(e_1)-x_2w'(e_2)<0$ as required. □

We now prove the following important consequences of this result.

Corollary 8.5. *1. The representation $\rho\colon W\to\mathrm{Aut}(\Gamma)$ is faithful.*

2. Every real root is either positive or negative, and $\Phi^{\mathrm{re}}=-\Phi^{\mathrm{re}}$.

3. The length of $w\in W$ can be interpreted geometrically as

$$\ell(w)=|\{a\in\Phi^{\mathrm{re}}:a>0>w(a)\}|.$$

Proof. 1. If $w\neq 1$, then $\ell(ws_i)<\ell(w)$ for some i, so $w(e_i)<0$ and hence $\rho(w)\neq 1$.

2. Every real root is of the form $w(e_i)$ for some w and i. Now $\ell(ws_i)=\ell(w)\pm 1$, so $w(e_i)$ is either positive or negative.

3. Set $X(w):=\{a\in\Phi^{\mathrm{re}}:a>0>w(a)\}$. Clearly $X(1)=\emptyset$. Also, if $a=v(e_j)\in X(s_i)$, then necessarily $a=me_i$, so $e_j=mv^{-1}(e_i)$, and hence $m=1$. Thus $X(s_i)=\{e_i\}$. In general, write $w=w's_i$ with $\ell(w')<\ell(w)$. Then $X(w)=s_i(X(w'))\cup\{e_i\}$, which by induction has cardinality $\ell(w')+1=\ell(w)$. □

In particular, given a Coxeter element c, we see that the p_i are precisely the real roots a such that $a>0>c(a)$.

8.5 Positive semidefinite Cartan data

Let B be a Cartan datum, and $(-,-)$ the corresponding symmetric bilinear form on the root lattice Γ. We say that $x\in\Gamma$ is sincere if it has full support, indivisible provided $x=my$ implies $m=\pm 1$, and radical if $(x,y)=0$ for all y.

The following result characterises the connected Cartan data of affine type.

Proposition 8.6. *Let B be a connected Cartan datum, with Dynkin diagram Δ. Then B is affine if and only if there exists a positive radical element in Γ.*

In this case, the sublattice of radical elements is precisely $\mathbb{Z}\delta$ for some positive, sincere and indivisible element δ. Moreover, every proper subdiagram of Δ is finite, and every Dynkin diagram properly containing Δ is wild.

Proof. We begin with the following observation. Let $x\in\Gamma$ be non-zero such that $(x,x)=0$. If $(x,e_i)\neq 0$ for some i, then

$$(mx+e_i,mx+e_i)=2d_i+2m(x,e_i)\quad\text{for all } m\in\mathbb{Z},$$

so the form must be indefinite.

Suppose that B is positive semidefinite, but not positive definite. Then there exists a non-zero $x\in\Gamma$ with $(x,x)=0$. Moreover, x must be both sincere and radical (otherwise $(x,e_i)\neq 0$ for some i and the form is indefinite, a contradiction). In particular, we see that every proper subdiagram of Δ is positive definite, and every Dynkin diagram properly containing Δ is indefinite.

Write $x = x_+ - x_-$ with $x_+, x_- \geq 0$ and having disjoint supports. Expanding out $(x, x) = 0$ yields

$$2(x_+, x_-) = (x_+, x_+) + (x_-, x_-).$$

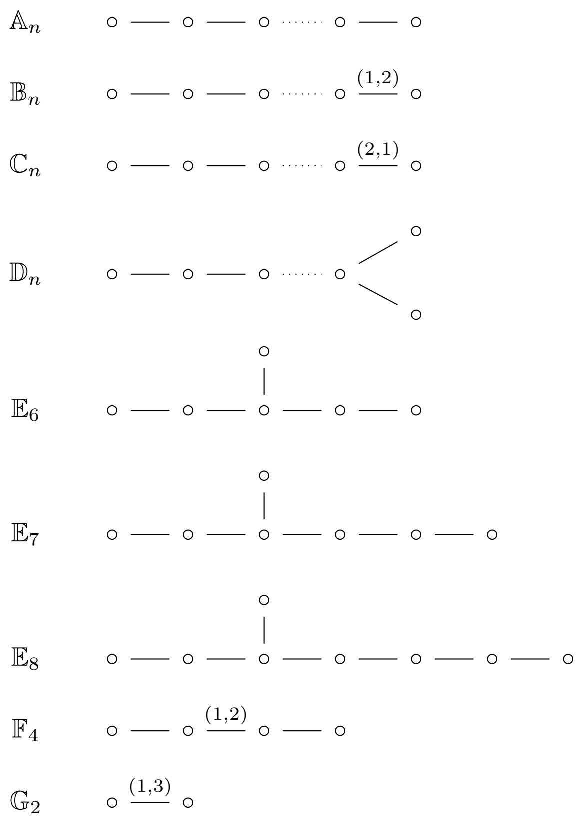
The left hand side is non-positive, since $x_\pm$ have disjoint supports, whereas the right hand side is non-negative since B is positive definite. Thus $(x_+, x_-) = 0$, and since B is connected and x is sincere, one of $x_\pm$ must be zero. Rescaling we may therefore assume that x is positive, sincere and indivisible.

Conversely, suppose we have a positive radical element $x \in \Gamma$. Since B is connected we know that x is sincere. Now note that $\sum_j b_{ij} x_j = 0$, and $b_{ij} x_i x_j \leq 0$ for $i \neq j$. Thus for all $y \in \Gamma$ we have

$$\begin{aligned} -\sum_{i<j} b_{ij} x_i x_j \Big(\frac{y_i}{x_i} - \frac{y_j}{x_j}\Big) &= -\sum_{i \neq j} b_{ij} x_j \frac{y_i^2}{x_i} + \sum_{i \neq j} b_{ij} y_i y_j \\ &= \sum_i b_{ii} y_i^2 + \sum_{i \neq j} y_i b_{ij} y_j = (y, y). \end{aligned}$$

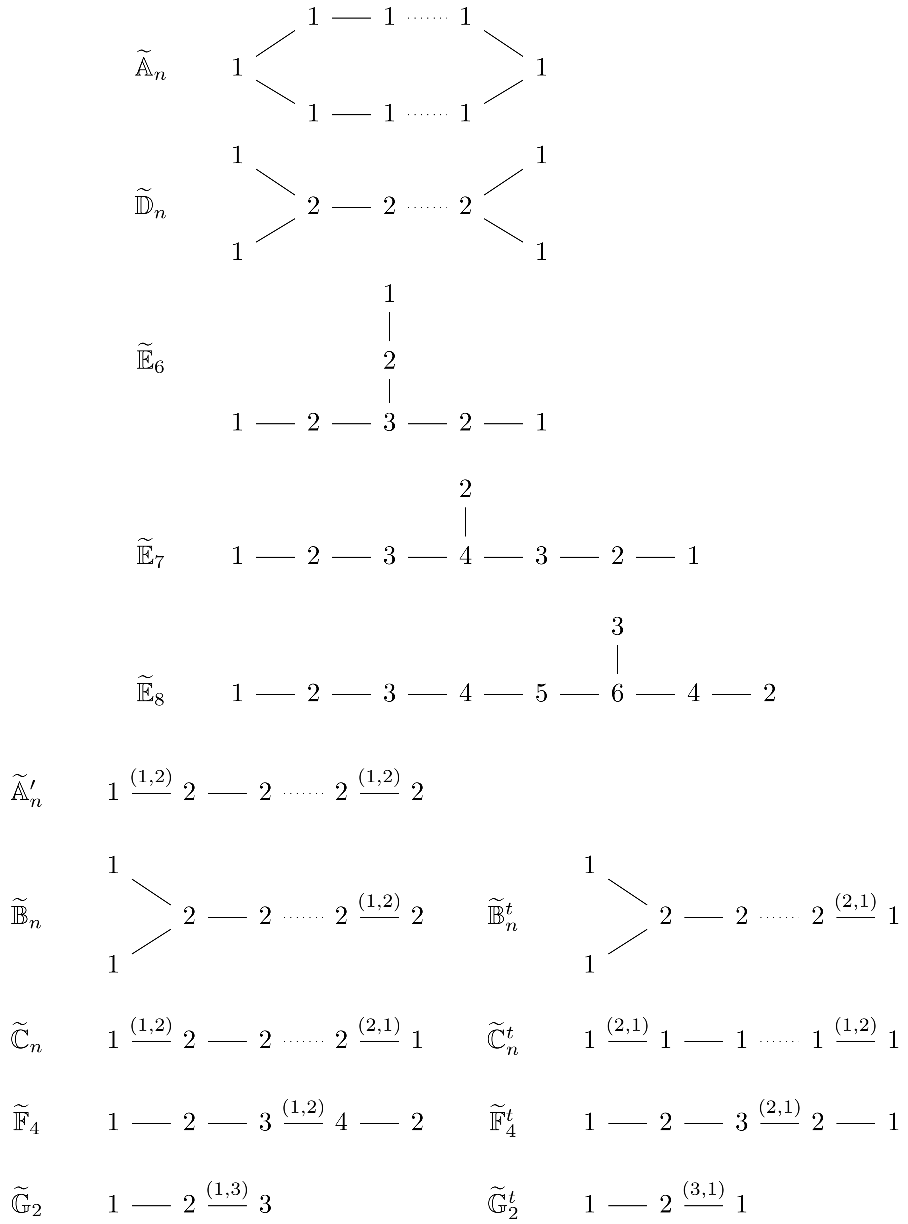
Thus the bilinear form is positive semidefinite, and moreover $(y, y) = 0$ if and only if y is proportional to x. $\square$

8.6 The Dynkin diagrams of finite type



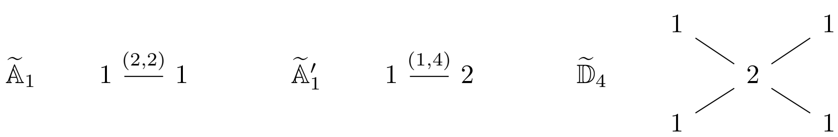
Note. The subscript denotes the number of vertices.

8.7 The Dynkin diagrams of affine type

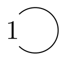


Notes. The subscript is one less than the number of vertices, and the vertex labels give the element $\delta \in \Gamma$.

We remark that there are a few degenerate cases.



We also sometimes allow the Dynkin diagram



although this corresponds to the zero matrix, so is an example of a symmetrisable Borcherds matrix.

8.8 The classification theorem

Theorem 8.7. *The preceding lists classify the Dynkin diagrams of finite and affine type.*

Proof. We check by inspection that for each diagram on the affine list, the corresponding element δ is positive, sincere, indivisible and radical. Thus these diagrams are all of affine type. Moreover, each diagram on the finite list is a proper subdiagram of an affine diagram, and hence is indeed of finite type. It remains to show that every connected diagram not on one of the lists is of indefinite type, which is necessarily the case if it properly contains some diagram on the affine list.

Consider a valued edge $1 \overset{(a,b)}{\text{———}} 2$, and take positive integers d_1, d_2 such that $d_1 a = d_2 b$. Then for $x = x_1 e_1 + x_2 e_2 \in \Gamma$ we have

$$\tfrac{1}{2}(x,x) = d_1 x_1^2 - d_1 a x_1 x_2 + d_2 x_2^2.$$

The discriminant of this form is $d_1 d_2(ab-4)$, so the form is indefinite whenever $ab > 4$, whereas $ab = 4$ implies we have $\tilde{\mathbb{A}}_1$ or $\tilde{\mathbb{A}}_1'$, which are on the affine list.

Suppose next that Δ contains a valued edge with $ab = 3$, which is $\mathbb{G}_2$ on the finite list. If this is not all of Δ, then we must have a subdiagram of the form

$$\circ \overset{(a',b')}{\text{———}} \circ \overset{(1,3)}{\text{———}} \circ \quad \text{or} \quad \circ \overset{(a',b')}{\text{———}} \circ \overset{(3,1)}{\text{———}} \circ$$

Consider $x \in \Gamma$ given by $(1,2,3)$ in the first case, and $(1,2,1)$ in the second case. Then in both cases $\frac{1}{2}(x,x) = d_1(1-a') + d_2(1-b')$, so the form is indefinite unless $a' = b' = 1$, in which case we have $\tilde{\mathbb{G}}_2$ or $\tilde{\mathbb{G}}_2^t$, which are on the affine list.

We may now assume that every valued edge satisfies $ab \leq 2$. If Δ contains two edge with $ab = 2$, then it contains one of $\tilde{\mathbb{A}}_n'$, $\tilde{\mathbb{C}}_n$ or $\tilde{\mathbb{C}}_n^t$ with $n \geq 2$ as a subdiagram, and these are all on the affine list. Suppose Δ contains precisely one edge with $ab = 2$. Then this edge cannot lie on a cycle in Δ, since otherwise the corresponding matrix is not symmetrisable. If Δ has a vertex of valency at least three, then it contains one of $\tilde{\mathbb{B}}_n$ or $\tilde{\mathbb{B}}_n^t$ as a subdiagram, and again these lie on the affine list. Assume therefore that every vertex has valency at most two. Then Δ is either of type $\mathbb{B}_n$, $\mathbb{C}_n$ or $\mathbb{F}_4$, which are all on the finite list, or else is contains a subdiagram of type $\tilde{\mathbb{F}}_4$ or $\tilde{\mathbb{F}}_4^t$, which are both on the affine list.

Finally, we may assume that all valued edges satisfy $ab = 1$. If Δ contains a cycle, then it contains some $\tilde{\mathbb{A}}_n$ with $n \geq 2$. Assume therefore that Δ is contains no cycles (so is a tree). If it contains a vertex of valency at least four, then it contains $\tilde{\mathbb{D}}_4$, whereas if it contains two vertices of valency three, then it contains some $\tilde{\mathbb{D}}_n$ with $n \geq 5$. If it contains precisely one vertex of valency three, then either it is equal to $\mathbb{D}_n$ or $\mathbb{E}_{6,7,8}$, or else it contains $\tilde{\mathbb{E}}_{6,7,8}$. The only remaining case is a tree where every vertex has valency at most two, which is of type $\mathbb{A}_n$. □

8.9 Cartan data of finite type

The following theorem collects a number of useful characterisations of finite type.

Theorem 8.8. *The following are equivalent for a Cartan datum B.*

1. *B is of finite type.*
2. *The set of real roots Φ^{re} is finite.*
3. *The Weyl group is finite.*
4. *Some (and hence all) Coxeter element $c \in W$ has finite order.*

Proof. $1 \Rightarrow 2$. Let B be of finite type. Then the bilinear form is positive definite, so as in the proof of Gabriel's Theorem we know that there are only finitely many elements $a \in \Gamma$ with $(a, a) \leq 2d$. Taking $d = \max\{d_i\}$ we see that there are only finitely many real roots.

$2 \Rightarrow 3$. The Weyl group acts faithfully on Γ, which is spanned by Φ^{re}. Thus W acts faithfully on the finite set Φ^{re}, so W is finite.

$3 \Rightarrow 4$. If W is finite, then every element has finite order.

$4 \Rightarrow 1$. There is a purely combinatorial proof of this, but it uses the classification theorem. An alternative approach is to use representation theory.

Given a Coxeter element c, we have the corresponding generalised Cartan lattice, which we can realise as the Grothendieck group of a finite dimensional hereditary algebra R. Now if c has finite order h, then $c^{h-1}(q_i) = -p_i < 0$, so every injective must be preprojective by Proposition 7.22. Similarly every projective is postinjective, so by Gabriel's Theorem R has finite representation type and the bilinear form is positive definite. □

8.10 Gabriel's Theorem revisited

Lemma 8.9. *Let B be a Cartan datum which is either finite or rank two, and let c be any Coxeter element. Then the set of real roots is precisely the c-orbits of the p_i and q_i.*

Proof. Suppose first that B is of finite type, and suppose that a is a positive real root which is not in the c-orbit of any p_i or q_j. Then for each integer r we know that $c^r(a)$ is again a positive real root, so if c has order h, then $x := a + c(a) + \cdots + c^{h-1}(a)$ is positive and c-invariant. It follows that $\langle x, x\rangle = -\langle x, c(x)\rangle = -\langle x, x\rangle$, so $\langle x, x\rangle = 0$. Hence $(x, x) = 0$, a contradiction.

Now assume that B has rank two. Then taking $c = s_2 s_1$, we have $p_1 = e_1$, $p_2 = s_1(e_2)$, $q_1 = s_2(e_1)$ and $q_2 = e_2$. Now any Weyl group element is uniquely of the form c^r or $c^r s_2$ for some $r \in \mathbb{Z}$, in which case $c^r(e_1) = c^r(p_1)$ and $c^r s_2(e_1) = c^r(q_1)$. Similarly, any Weyl group is uniquely of the form c^r or $c^r s_1$ for some $r \in \mathbb{Z}$, in which case $c^r(e_2) = c^r(q_2)$ and $c^r s_1(e_2) = c^r(p_2)$. □

Proposition 8.10. *Let R be a finite dimensional hereditary algebra, which is either of finite representation type or rank two. Then every exceptional R-module is either preprojective or postinjective, and the map $X \mapsto [X]$ induces a bijection between the isomorphism classes of exceptional modules and the positive real roots.*

Proof. We know from Theorem 7.23 that there is a bijection between the isomorphism classes of indecomposable preprojectives and the set of positive real roots of the form $c^{-r}(p_i)$, and dually for the postinjectives. Moreover, each indecomposable preprojective or postinjective is exceptional. When R has finite representation type or rank two, then the bilinear form is either positive definite (by Gabriel's Theorem) or rank two, so the previous lemma tells us that every real root is of this form.

It remains to show that when R has finite representation type or rank two, then every exceptional module is either preprojective or postinjective. When R has finite representation type, this follows from Gabriel's Theorem.

Suppose therefore that R has rank two, and let X be an exceptional module. We first show that $x := [X]$ is a positive real root.

By assumption, $\mathrm{End}_R(X)$ is a division algebra of dimension $\langle x,x\rangle$ over k, and for any R-module Y both $\mathrm{Hom}_R(X,Y)$ and $\mathrm{Ext}^1_R(X,Y)$ are naturally right modules over $\mathrm{End}_R(X)$. It follows that $\langle x,x\rangle$ divides $\langle x,y\rangle$ for all $y>0$, and hence $\langle x,x\rangle$ divides $\langle x,y\rangle$ for all $y\in\Gamma$. Similarly $\langle x,x\rangle$ divides $\langle y,x\rangle$ for all $y\in\Gamma$.

Now let $x\in\Gamma$ be any element with $\langle x,x\rangle>0$, and dividing both $\langle x,y\rangle$ and $\langle y,x\rangle$ for all $y\in\Gamma$. We first observe that x is either positive or negative. For, suppose $x=me_1-ne_2$ with $m,n>0$, and assume as usual that $\langle e_1,e_2\rangle=0$. Then $\langle x,e_2\rangle=-n\langle e_2,e_2\rangle=-nd_2$ and $\langle e_1,x\rangle=m\langle e_1,e_1\rangle=md_1$, and these are both divisible by the strictly larger number $\langle x,x\rangle=m^2d_1+n^2d_2+mnb$, where $b=-\langle e_2,e_1\rangle>0$, a contradiction.

Next observe that if x satisfies these conditions, then so too does every $w(x)$ for $w\in W$. For, we have $\langle s_i(x),y\rangle=\langle x,y\rangle+\frac{1}{d_i}(x,e_i)\langle e_i,y\rangle$. Since d_i divides $\langle e_i,y\rangle$, we see that $\langle x,x\rangle$ divides the right hand side, and as $\langle s_i(x),s_i(x)\rangle=\langle x,x\rangle$, the claim follows.

Assume now that $x>0$ satisfies these conditions, and is the minimal positive element in its W-orbit. If $(x,e_i)\leq 0$ for $i=1,2$, then $(x,x)\leq 0$ a contradiction. Thus we must have $(x,e_i)>0$ for some i, in which case $s_i(x)<x$, so by minimality $s_i(x)<0$, and hence $x=me_i$. The divisibility conditions then give $m=1$, so $x=e_i$ is a real root. The only other possibility is if $(x,e_i)\leq 0$ for $i=1,2$, which implies $(x,x)\leq 0$, a contradiction.

Thus X exceptional implies $[X]$ is a positive real root, so of the form $c^{-r}(p_i)$ or $c^r(q_j)$, and hence X is preprojective or postinjective by Proposition 7.22. □

8.11 Conjugacy classes of Coxeter elements

Let Δ be a Dynkin diagram, and W its Weyl group. Given a vertex i, write Δ' for the Dynkin diagram obtained by deleting vertex i, and let W' be theWeyl group of Δ'. Finally let W_i to be the subgroup of W generated by all simple reflections s_j for $j\neq i$.

Lemma 8.11. *With the notation as above, there is a natural isomorphism $W_i \xrightarrow{\sim} W'$ identifying the simple reflections s_j.*

Proof. Since the generators s_j for W_i satisfy the necessary relations, we see that there is a surjective group homomorphism $W'\twoheadrightarrow W_i$ identifying the s_j.

To construct the inverse map, choose a Cartan datum of type Δ, yielding the root lattice Γ with symmetric bilinear form $(-,-)$. Let $\Gamma_i\leq\Gamma$ be the sublattice spanned by the simple roots e_j for $j\neq i$. Then the restriction of $(-,-)$ to Γ_i

is a Cartan datum of type Δ', so we can identify Γ_i with a root lattice Γ' of Δ'. In particular, there is a faithful representation $W' \to \mathrm{Aut}(\Gamma_i)$, so we can identify W' with its image.

Now the simple reflections s_j for $j \neq i$ preserve the sublattice Γ_i, so we have a representation $W_i \to \mathrm{Aut}(\Gamma_i)$. This identifies the simple reflactions s_j to s_j, so yields a surjective group homomorphism $W_i \to W'$, inverse to the map $W' \to W_i$ given above. □

Let $c = s_{i_n} \cdots s_{i_1}$ be a Coxeter element in W. Suppose the simple reflections s_{i_r} and $s_{i_{r+1}}$ commute; equivalently there is no valued edge in Δ connecting the vertices i_r and i_{r+1}, Then swapping them leaves the Coxeter element c unchanged. Such a move will be called a swap. Similarly conjugating by s_{i_1} yields the Coxeter element $s_{i_1} s_{i_n} \cdot s_{i_2}$. Such a move (or its inverse) will be called a rotation.

Theorem 8.12. *If the Dynkin diagram is a tree, then any two Coxeter elements are related by a sequence of swaps and rotations.*

Suppose instead c is a Coxeter element for the Dynkin diagram $\tilde{\mathbb{A}}_{n-1}$, *so the cycle* $(1, 2, \ldots, n)$. *Let p be the number of anticlockwise arrows in the corresponding orientation. Then c is related by a sequence of swaps and rotations to the Coxeter element* $(s_{p+1} \cdots s_n)(s_p \cdots s_1)$.

Proof. Suppose the Dynkin diagram is a union of trees, and let c be any Coxeter element. Let vertex 1 be a leaf, so it has a unique neighbour, say vertex 2. Since s_1 commutes with all s_i for $i \geq 3$, we can apply a sequence of swaps to transform c into either $s_1 c'$ or $c' s_1$, where c' is a Coxeter element for $W' = \langle s_2, \ldots, s_n \rangle$. Rotating if necessary we have transformed c into $c' s_1$.

By the lemma, we know that W' is the Weyl group of the Dynkin diagram $\Delta - \{1\}$, which is again a union of trees. By induction we can transform c' to $s_n \cdots s_2$ by a sequence of swaps and rotations. We now show how to lift this to a sequence of swaps and rotations transforming $c' s_1$ to $s_n \cdots s_1$. From this it will follow that all Coxeter elements are conjugate, and can be transformed into $s_n \cdots s_1$ using a sequence of swaps and rotations.

Consider a Coxeter element $c'' = s_{j_n} \cdots s_{j_2}$ in W', and the Coxeter element $c'' s_1$ in W. Any swap on c'' yields a swap on $c'' s_1$, so consider a rotation, say conjugating c'' by s_{j_2}. Rotating $c'' s_1$ twice yields $s_{j_2} s_1 s_{j_n} \cdots s_{j_3}$, which we can transform using swaps and possibly a rotation as above into $s_{j_2} \cdots s_{j_n} s_1$. This proves the result for trees.

Now suppose that Δ is the cycle $(1, 2, \ldots, n)$, so affine of type $\tilde{\mathbb{A}}_{n-1}$. We claim that we can transform c into $(s_p \cdots s_n)(s_{p-1} \cdots s_1)$ for some p using swaps and rotations. Define $\Delta_{>i}$ to be the subdiagram on vertices $i+1, \ldots, n$, a tree of type $\mathbb{A}_{n-i}$.

We begin by rotating until we have $c' s_1$ for some Coxeter element c' for $\Delta_{>1}$. Note that vertex 2 is a leaf for this subdiagram, so we can use swaps to transform c' into $c'' s_2$ or $s_2 c''$. Continuing in this way, we can use swaps to transform $c' s_1$ into $(s_i \cdots s_j) c'' s_{j+1} (s_{i-1} \cdots s_1)$ for some $2 \leq i \leq j < n$, where now c'' is a Coxeter element for $\Delta_{>j+1}$. If $j = n-1$, then $c'' = 1$ and we are done. Otherwise we use swaps to transform this into $c''(s_i s_{i-1} \cdots s_1)(s_{i+1} \cdots s_j s_{j+1})$, and then rotate to get $(s_{i+1} \cdots s_{j+1}) c'' (s_i \cdots s_1)$. Now vertex $j+2$ is a leaf for $\Delta_{>j+1}$, so we can repeat. In this way we can transform c into a Coxeter element of the required form.

Now, any swap leaves the induced orientation the same, whereas a rotation corresponds to choosing a sink or source and reflecting both the incident arrows. Thus this does not affect the number of anticlockwise arrows. Now observe that the orientation corresponding to $(s_{p+1} \cdots s_n)(s_p \cdots s_1)$ is given by making vertex 1 the unique sink and vertex $p+1$ the unique source, so has p anticlockwise arrows (and $n-p$ clockwise arrows). $\square$

In particular, the characteristic polynomial of c acting on the root lattice Γ is an invariant of Δ when Δ is a tree.

9 Tame representation type

We say that an indecomposable, finite dimensional hereditary algebra has tame representation type if its Grothendieck group is of affine type. The terminology comes from the fact that in this case, even though there are infinitely many indecomposable modules, we can classify them explicitly.

9.1 Regular modules and the defect

Let R be a finite dimensional hereditary algebra. A full subcategory $\mathcal{T} \subset \operatorname{mod} R$ is said to be thick provided it is closed under direct summands and satisfies the 2-out-of-3 property; that is, given a short exact sequence $0 \to X \to Y \to Z \to 0$, if two of X, Y, Z are in $\mathcal{T}$, then so too is the third.

It is clear that intersections of thick subcategories are again thick. In particular, given a collection $\mathcal{X}$ of R-modules we can define $\operatorname{thick}(\mathcal{X})$ to be the smallest thick subcategory containing $\mathcal{X}$.

A thick subcategory closed under images is called a thick abelian. In this case we can talk about $\mathcal{T}$-simple modules, which are those modules $X \in \mathcal{T}$ having no proper submodule in $\mathcal{T}$.

Lemma 9.1. *For an hereditary algebra, every thick subcategory is thick abelian.*

Proof. Consider a map $f \colon X \to Y$ with X, Y in a thick subcategory $\mathcal{T}$. By Proposition 6.1 (6) we know that there is a short exact sequence $0 \to X \to E \oplus \operatorname{Im}(f) \to Y \to 0$, so $E \oplus \operatorname{Im}(f)$, and hence also $\operatorname{Im}(f)$ lies in $\mathcal{T}$. □

Lemma 9.2. *Let B be a Cartan datum of affine type. Then every element of the Weyl group has finite order on $\Gamma/\mathbb{Z}\delta$.*

Proof. We know that $(\delta, x) = 0$ for all $x \in \Gamma$, and $(x, x) = 0$ if and only if $x \in \mathbb{Z}\delta$. Thus the bilinear form restricts to a positive definite bilinear form on $\Gamma/\mathbb{Z}\delta$; also $s_i(\delta) = \delta$, so s_i induces an action on $\Gamma/\mathbb{Z}\delta$. Hence we have a representation $W \to \operatorname{Aut}(\Gamma/\mathbb{Z}\delta)$. Now, $(w(x), w(x)) = (x, x)$, so by the Gram-Schmidt algorithm we see that there are only finitely many $x \in \Gamma/\mathbb{Z}\delta$ for which $(x, x) \leq r$ for any r. In particular, setting r to be the maximum of (e_i, e_i), we see that each orbit $w(e_i)$ is finite in $\Gamma/\mathbb{Z}\delta$, and hence w itself has finite order $\Gamma/\mathbb{Z}\delta$. □

Proposition 9.3. *Let R be a tame hereditary algebra, and let $\delta \in \Gamma$ be the minimal positive radical element. Then an indecomposable module X is preprojective, regular, or postinjective according to whether $\langle \delta, X \rangle$ is negative, zero, or positive.*

Proof. We first observe that $(\delta, e_i) = 0$, so $s_i(\delta) = \delta$ for all i. In particular, $c(\delta) = \delta$. Then $\langle \delta, x \rangle = \langle \delta, c^r(x) \rangle$, so this number is constant on c-orbits.

Now, writing $\delta = \sum_i \delta_i e_i$, we have $\langle \delta, q_i \rangle = \delta_i d_i > 0$, and hence also $\langle \delta, p_i \rangle = \langle \delta, -c(q_i) \rangle = -\delta_i d_i < 0$. This proves the result for preprojectives and postinjectives. It remains to prove that all regular modules X satisfy $\langle \delta, X \rangle = 0$.

Suppose X is a regular module, and set $x := [X]$. Then $c^r(x) = [\tau^r X] > 0$ for all $r \in \mathbb{Z}$. By the previous lemma we know that c has finite order h modulo δ, so $c^h(x) = x + m\delta$ for some $m \in \mathbb{Z}$. Then $c^{rh}(x) = x + rm\delta > 0$ for all $r \in \mathbb{Z}$, so $m = 0$ and $c^h(x) = x$. Set $y := x + c(x) + \cdots + c^{h-1}(x) > 0$. Then $c(y) = y$, so as in the proof of Lemma 8.9, y is a radical vector, hence a multiple of δ. It follows that $h\langle \delta, x\rangle = \langle \delta, y\rangle = 0$. ☐

We have shown that if R is tame, then the map $\Gamma \to \mathbb{Z}$, $x \mapsto \langle \delta, x\rangle$, is non-zero. In particular, its image is $r\mathbb{Z}$ for some $r > 0$, and so we define the defect to be the normalised linear form $\partial(x) := \frac{1}{r}\langle \delta, x\rangle$.

We usually simplify notation and refer to the defect $\partial(X)$ of a module X, instead of just the defect of its class $\partial([X])$. Note that $\Gamma = \mathrm{Ker}(\partial) \oplus \mathbb{Z}[X]$ for any module X of defect ± 1, that $\partial(\delta) = 0$, and that $\langle x, \delta\rangle = -\langle \delta, x\rangle$.

Theorem 9.4. *Let R be an indecomposable finite dimensional hereditary algebra. Then R is tame if and only if the regular modules form a thick abelian subcategory.*

In this case, every regular-simple module S has finite order under τ. Moreover, if S has order p, then the classes $[S], c[S], \dots, c^{p-2}[S]$ are linearly independent in Γ, and their span is a generalised Cartan lattice of type $\mathbb{A}_{p-1}$.

Proof. Suppose R is of affine type. We know that the class of regular modules is closed under extensions and images, so in particular direct summands of regular modules are again regular. Now suppose that $f\colon X \twoheadrightarrow Y$ is an epimorphism of regular modules. Then its kernel has no positinjective summand and has defect zero, hence is again regular. Similarly the cokernel of a monomorphism between regulars is again regular, so the regulars form a thick abelian subcategory.

Now suppose that the regular modules form a thick abelian subcategory. We first observe that the Auslander-Reiten translate is an exact functor on the subcategory of regular modules. For, suppose that $0 \to X \to Y \to Z \to 0$ is an exact sequence of regular modules. Then $\tau^- X \to \tau^- Y \to \tau^- Z \to 0$ is again exact, and computing the classes in the Grothendieck group we have

$$[\tau^- X] + [\tau^- Z] - [\tau^- Y] = c^-[X] + c^-[Z] - c^-[Y] = c^-\big([X] + [Z] - [Y]\big) = 0.$$

Thus the map $\tau^- X \to \tau^- Y$ must be injective, so we have an exact sequence $0 \to \tau^- X \to \tau^- Y \to \tau^- Z \to 0$. Similarly $0 \to \tau X \to \tau Y \to \tau Z \to 0$ is exact.

It follows that if S is regular-simple, then so too is $\tau^r S$ for all r. Now let S' be another regular simple. Then by the Auslander-Reiten Formula $\mathrm{Ext}^1_R(S, S') \cong D\,\mathrm{Hom}_R(S', \tau S)$. Since both S' and τS are regular simple, this is non-zero if and only if $S' \cong \tau S$.

We next want to show that $\tau^p S \cong S$ for some $p > 0$. Suppose therefore that $S, \tau S, \dots, \tau^r S$ are pairwise non-isomorphic. Then for $0 \le i, j < r$ we have $\langle \tau^i S, \tau^j S\rangle = \langle S, S\rangle\big(\delta_{ij} - \delta_{i+1j}\big)$. It follows that the classes $[S], c[S], \dots, c^{r-1}[S]$ are linearly independent in Γ, so $r \le n$, and their span is a generalised Cartan lattice of type $\mathbb{A}_r$.

Finally, if $\tau^p S \cong S$, then $x := [S] + c[S] + \cdots + c^{p-1}[S] > 0$ is c-invariant, hence is positive and radical. It follows from Proposition 8.6 that R has affine type, and that x is a multiple of δ. ☐

Let S_x for $x \in \mathbb{X}$ be representatives for the orbits of regular-simple modules, and let p_x be the order of S_x under τ.

Lemma 9.5. *Let R be an indecomposable, finite dimensional, tame hereditary algebra of rank n. Then $\sum_{\mathbb{X}}(p_x - 1) \le n - 2$.*

Proof. Let S be a regular-simple of order p under τ. We have just seen that the classes $[S], c[S], \dots, c^{p-2}[S]$ are linearly independent in Γ, and their span is a generalised Cartan lattice of type $\mathbb{A}_{p-1}$.

We can repeat this construction for all τ-orbits of regular-simples, and since there are no homomorphisms or extensions between regular-simples in distinct τ-orbits, the bilinear form has block diagonal form on their span, with one block of type $\mathbb{A}_{p-1}$ for each orbit of size p. In particular, this yields a sublattice $\Gamma' \leq \Gamma$ of rank $\sum_x (p_x - 1)$ on which the bilinear form is positive definite.

Now, since Γ' has a basis given by regular-simples, it is containined inside $\mathrm{Ker}(\partial)$, and since the bilinear form is positive definite on Γ', it does not contain δ. Thus the rank of Γ' is at most $n-2$. $\square$

Our first task is to prove that we always have equality, so $\sum_x (p_x - 1) = n - 2$ for all tame hereditary algebras R.

9.2 Tame homogeneous algebras

Let R be a tame hereditary algebra. We will say that R is homogeneous if $p_x = 1$ for all $x \in \mathbb{X}$. Clearly if R has rank 2, then since $\sum_x (p_x - 1) \leq n - 2$ it must be homogeneous. In this section we will prove the converse.

Lemma 9.6. *Let R be a tame hereditary algebra, and P and Q two preprojective modules. If $\partial(P) = -1$, then any non-zero homomorphism $P \to Q$ is necessarily injective, and the cokernel has no postinjective summand.*

Proof. Let $f\colon P \to Q$ be non-zero. We know that the image is non-zero, and both the image and kernel are preprojective. Then $\partial(\mathrm{Ker}(f)) = \partial(P) - \partial(\mathrm{Im}(f)) \geq 0$, so $\mathrm{Ker}(f) = 0$ and f is injective. Now suppose that I is a non-zero postinjective summand of the cokernel. Taking the pull-back yields an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & E & \longrightarrow & I & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{f} & Q & \longrightarrow & \mathrm{Coker}(f) & \longrightarrow & 0.
\end{array}$$

Thus $E \rightarrowtail Q$ is a submodule, so is preprojective, but has defect $\partial(E) = \partial(P) + \partial(I) \geq 0$, a contradiction. $\square$

Proposition 9.7. *Let R be a tame hereditary algebra which is homogeneous. Then $\mathrm{Ker}(\partial) = \mathbb{Z}\delta$ and R has rank two.*

Proof. Let P be a (necessarily indecomposable) projective module of defect -1. If S is a regular-simple, then R homogeneous implies that $[S] = m\delta$ for some m. Let $\Gamma' \leq \Gamma$ be the sublattice spanned by $[P]$ together with the classes of all regular-simple modules. We claim that $\Gamma' = \Gamma$.

Let Q be any non-zero preprojective module. Since $\partial(Q) < 0$ we know that there is a non-split short exact sequence $0 \to Q \to E \to S \to 0$ for any regular-simple S. If S' is a non-zero regular-simple submodule of E, then the composition $S' \to E \to S$ is either zero or an isomorphism, and if it is zero, then S' factors through the preprojective Q, a contradiction. Thus $S' \to S$ is an isomorphism and the sequence splits, a contradiction. We conclude that E is preprojective.

Next, since $\langle P, S\rangle > 0$, there is a non-zero homomorphism $P \to S$, which we can lift to E since P is projective. By the previous lemma the map $P \to E$ is injective and the cokernel C has no postinjective summands. It follows that $[Q] = [P] + [C] - [S]$, and since $\partial(C) = \partial(Q) + 1$ we know by induction on the defect that $[C] \in \Gamma'$. (If $\partial(C) = 0$, then it is regular, so lies in Γ' by assumption.) Using that the classes of the indecomposable projectives form a basis for Γ, it follows that $\Gamma' = \Gamma$ as claimed.

Finally, since $[S] \in \mathbb{Z}\delta$ for each regular simple S, we see that Γ is spanned by $[P]$ and δ, and hence has rank two. □

9.3 Universal homomorphisms and extensions

Let R be an algebra, and X a finite dimensional R-module. We write $\mathrm{add}(X)$ for the class of modules which are direct summands of some X^r.

Let M be a finite dimensional R-module. A left $\mathrm{add}(X)$-approximation of M is a map $\lambda_M \in \mathrm{Hom}_R(M, X_M)$ such that $X_M \in \mathrm{add}(X)$ and composition with λ_M is onto

$$\lambda_M^*\colon \mathrm{Hom}_R(X_M, X) \twoheadrightarrow \mathrm{Hom}_R(M, X).$$

A universal $\mathrm{add}(X)$-extension of M is an element $\eta_M \in \mathrm{Ext}^1_R(M, X^1_M)$ such that $X^1_M \in \mathrm{add}(X)$ and the push-out map is onto

$$\eta_M^*\colon \mathrm{Hom}_R(X^1_M, X) \twoheadrightarrow \mathrm{Ext}^1_R(M, X).$$

Dually we have the notion of right $\mathrm{add}(X)$-approximation $\rho^M \in \mathrm{Hom}_R(X^M, M)$ of M and a universal $\mathrm{add}(X)$-coextension $\eta^M \in \mathrm{Ext}^1_R(X^M_1, M)$ of M.

Lemma 9.8. *Left* $\mathrm{add}(X)$*-approximations of M exist, as do universal* $\mathrm{add}(X)$*-extensions of M.*

Dually, right approximations and universal coextensions exist.

Proof. Let $f_1, \dots, f_r$ generate $\mathrm{Hom}_R(M, X)$ as a left $\mathrm{End}_R(X)$-module, and consider $f = (f_i)\colon M \to X^r$. If g is any homomorphism $M \to X$, then there exist $\xi_i \in \mathrm{End}_R(X)$ such that $g = \sum_i \xi_i f_i$. Thus $g = \xi f$ where $\xi = (\xi_i)\colon X^r \to X$, so g factors through f.

Next let $\eta_1, \dots, \eta_s$ generate $\mathrm{Ext}^1_R(M, X)$ as a left $\mathrm{End}_R(X)$-module. Then, given any other $\zeta \in \mathrm{Ext}^1_R(M, X)$ there exist $\xi_i \in \mathrm{End}_R(X)$ such that $\zeta = \sum_i \xi_i \eta_i$. Let $\nabla\colon X^s \to X$ and $\Delta\colon M \to M^s$ be the diagonal maps. Then the construction of the Baer sum gives $\zeta = \nabla(\bigoplus_i \xi_i\eta_i)\Delta$. Since taking pull-backs and push-outs commute, we can write this as $\zeta = \xi\eta$, where $\xi = \nabla(\bigoplus_i \xi_i) = (\xi_i)\colon X^r \to X$ and $\eta = (\bigoplus_i \eta_i) \in \mathrm{Ext}^1_R(X^s, M)$. Thus η is a universal $\mathrm{add}(X)$-extension of M. □

9.4 Rigid modules

In this section R will be a finite dimensional hereditary algebra. Recall that a finite dimensional R-module E is exceptional provided $\mathrm{End}_R(E)$ is a division algebra and $\mathrm{Ext}^1_R(E, E) = 0$. In general we say that a finite dimensional R-module E is rigid provided $\mathrm{Ext}^1_R(E, E) = 0$.

Lemma 9.9 (Happel-Ringel)**.** *Let X and Y be indecomposable modules such that $\mathrm{Ext}^1_R(Y,X)=0$. Then any non-zero homomorphism $X^r\to Y$ is either injective or surjective. In particular, each indecomposable rigid module is exceptional.*

Proof. Let $f\colon X\to Y$ be non-zero. Then by Proposition 6.1 (6) we have a short exact sequence $0\to X\to \mathrm{Im}(f)\oplus E\to Y\to 0$, which is split by assumption. Thus $X\oplus Y\cong \mathrm{Im}(f)\oplus E$, so by the Krull-Remak-Schmidt Theorem we must have either $\mathrm{Im}(f)\cong Y$ and f is surjective, or else $\mathrm{Im}(f)\cong X$ and f is injective.

In particular, let E be indecomposable and rigid. Then any non-zero endomorphism is necessarily an isomorphism, so $\mathrm{End}_R(E)$ is a division algebra and E is exceptional. □

Lemma 9.10. *Let X,Y,Z be modules such that $[X]=[Y]$. If $\mathrm{Hom}_R(X,Z)=0=\mathrm{Ext}^1_R(Y,Z)$, then also $\mathrm{Hom}_R(Y,Z)=0=\mathrm{Ext}^1_R(X,Z)$.*

Proof. Under the assumptions we have

$$-\mathrm{Ext}^1_R(X,Z)=\langle X,Z\rangle=\langle Y,Z\rangle=\mathrm{Hom}_R(Y,Z).$$ □

Proposition 9.11. *Let X be rigid. Then $\mathrm{End}_R(X)$ is a triangular algebra. In other words, taking representatives for the indecomposable summands of X, we can order them as $X_1,\dots,X_r$ such that $\mathrm{Hom}_R(X_i,X_j)=0$ for $i>j$.*

Proof. We first observe that $\mathrm{Ext}^1_R(X_i,X_j)=0$ for all i,j. In particular, each X_i is exceptional, by the Happel-Ringel Lemma. Also, if we have a circuit of non-zero homomorphisms

$$X_{i_1}\to X_{i_2}\to\cdots\to X_{i_s}\to X_{i_1},$$

then each map is either injective or surjective. If they were all injective, then their composition would be a proper injective endomorphism of X_{i_1}, a contradiction. Similarly they cannot all be surjective. By rotating the circuit if necessary, we then obtain a subsequence $X_h\twoheadrightarrow X_i\rightarrowtail X_j$, whose composition is neither injective nor surjective, a contradiction. Hence we can order the X_i such that $\mathrm{Hom}_R(X_i,X_j)=0$ for $i>j$.

Thus, if $X\cong\bigoplus X_i^{a_i}$, then we can write $\mathrm{End}_R(X)$ as a matrix algebra $(\mathrm{Hom}_R(X_j^{a_j},X_i^{a_i}))$, which will be lower triangular with the semisimple rings $\mathbb{M}_{a_i}(\mathrm{End}_R(X_i))$ on the diagonal. □

The next result shows that rigid modules are completely determined by their classes in the Grothendieck group. Compare this to Proposition 7.22, which says that if X and P are indecomposable with P preprojective, then $X\cong P$ if and only if $[X]=[P]$.

Proposition 9.12 (Kerner)**.** *Let X and Y be rigid such that $[X]=[Y]$. Then $X\cong Y$.*

Proof. We first prove that $X\oplus Y$ is rigid. Take a left $\mathrm{add}(X)$-approximation $f\colon X^r\to Y$, and write I and C for its image and cokernel. Then $\mathrm{Hom}_R(X,C)=0=\mathrm{Ext}^1_R(Y,C)$. For, we have $\mathrm{Ext}^1_R(Y,Y)\twoheadrightarrow\mathrm{Ext}^1_R(Y,C)$, and these vanish since Y is rigid. Similarly $\mathrm{Ext}^1_R(X,I)=0$. By assumption the composition $\mathrm{Hom}_R(X,X^r)\to\mathrm{Hom}_R(X,I)\rightarrowtail\mathrm{Hom}_R(X,Y)$ is onto, so $\mathrm{Hom}_R(X,I)\cong$

$\mathrm{Hom}_R(X,Y)$. Thus, applying $\mathrm{Hom}_R(X,-)$ to the short exact sequence $0 \to I \to Y \to C \to 0$ we deduce that $\mathrm{Hom}_R(X,C) = 0$.

By the earlier lemma it follows that $\mathrm{Hom}_R(Y,C) = 0$, so $C = 0$ and f is onto, and then also $\mathrm{Ext}^1_R(X,Y) = 0$. Analogously we must have an epimorphism $Y^s \twoheadrightarrow X$, and so $\mathrm{Ext}^1_R(Y,X) = 0$.

Now, given an indecomposable summand X_1 of X, we have $\langle Y, X_1\rangle = \langle X, X_1\rangle > 0$, so there exists an indecomposable summand Y_1 of Y and a non-zero map $Y_1 \to X_1$. Similarly, there exists an indecomposable summand X_2 of X and a non-zero map $X_2 \to Y_1$. Continuing in this way we obtain an infinite chain of non-zero maps

$$\cdots \to Y_2 \to X_2 \to Y_1 \to X_1,$$

where X_i and Y_i are indecomposable summands of X and Y respectively. Since there are only finitely many such indecomposable summands, we must have a circuit, in which case all the intervening maps are isomorphisms by the previous proposition. In particular, X and Y have a common indecomposable summand. Removing this common summand yields rigid modules X', Y' such that $[X'] = [Y'] < [X] = [Y]$, and the result now follows by induction. $\square$

9.5 Perpendicular categories of rigid modules

Let R be a finite dimensional hereditary algebra, and X an R-module. We define the right perpendicular category

$$X^{\perp} := \{Y : \mathrm{Hom}_R(X,Y) = 0 = \mathrm{Ext}^1_R(X,Y)\}$$

and the left perpendicular category

$${}^{\perp}X := \{Y : \mathrm{Hom}_R(Y,X) = 0 = \mathrm{Ext}^1_R(Y,X)\}.$$

More generally, if $\mathcal{X}$ is a collection of R-modules, then we define $\mathcal{X}^{\perp}$ to be the intersection $\bigcap_{\mathcal{X}} X^{\perp}$, and similarly for the left perpendicular category ${}^{\perp}\mathcal{X}$.

Lemma 9.13. *The left and right perpendicular categories are thick abelian subcategories.*

Proof. We prove this for the right perpendicular category $X^{\perp}$ for a module X. The result for $\mathcal{X}^{\perp}$ follows by taking intersections, and the result for ${}^{\perp}\mathcal{X}$ is entirely analogous.

Clearly $X^{\perp}$ is closed under direct summands, so suppose we have a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules and apply $\mathrm{Hom}(X,-)$. Then we have the six term exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(X,A) \longrightarrow \mathrm{Hom}_R(X,B) \longrightarrow \mathrm{Hom}_R(X,C) \longrightarrow \mathrm{Ext}^1_R(X,A) \longrightarrow \mathrm{Ext}^1_R(X,B) \longrightarrow \mathrm{Ext}^1_R(X,C) \longrightarrow 0.$$

Thus if two of A,B,C lie in $X^{\perp}$, then so does the third, and hence $X^{\perp}$ is a thick subcategory. It is then thick abelian by Lemma 9.1. $\square$

Proposition 9.14. *Let X be a rigid module. Then every module M fits into a five term exact sequence*

$$0 \to M^1 \to X_0 \to M \to M^0 \to X_1 \to 0$$

with $X_0, X_1 \in \operatorname{thick}(X)$ and $M^0, M^1 \in X^\perp$.

Moreover, ${}^\perp(X^\perp) = \operatorname{thick}(X)$ and we have natural isomorphisms

$$\operatorname{Hom}_R(X', X_0) \xrightarrow{\sim} \operatorname{Hom}_R(X', M) \quad \text{for all } X' \in \operatorname{thick}(X)$$

and

$$\operatorname{Hom}_R(M^0, N) \xrightarrow{\sim} \operatorname{Hom}_R(M, N) \quad \text{for all } N \in X^\perp.$$

Proof. We begin by taking a universal $\operatorname{add}(X)$-coextension of M

$$0 \to M \to E \to X^M \to 0,$$

so the pull-back map $\operatorname{Hom}_R(X, X^M) \to \operatorname{Ext}^1_R(X, M)$ is onto. Since X is rigid it then follows that $\operatorname{Ext}^1_R(X, E) = 0$.

Next take a right $\operatorname{add}(X)$-approximation $X_E \to E$. Since R is hereditary and X is rigid, the image I of the composition $X_E \to E \to X^M$ is a direct summand of $X_E \oplus X^M$, so lies in $\operatorname{add}(X)$. It follows that the kernel K and cokernel X_1 lie in $\operatorname{thick}(X)$. We thus have the exact commutative diagram

$$\begin{CD}
0 @>>> K @>>> X_E @>>> I @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> M @>>> E @>>> X^M @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> N @>>> M^0 @>>> X_1 @>>> 0
\end{CD}$$

Applying $\operatorname{Hom}_R(X, -)$ to the middle row and column gives $\operatorname{Ext}^1_R(X, E) = 0$, and hence $M^0 \in X^\perp$.

Set J and L to be the kernel and image of $K \to M$, and observe that J is also the kernel of $X_E \to E$ by the Snake Lemma. In particular, $\operatorname{Ext}^1_R(X, J) = 0$. Take a left $\operatorname{add}(X)$-approximation $X_J \to J$ of J, say with cokernel M^1, and then form the push-out, giving an exact commutative diagram

$$\begin{CD}
@. X_J @= X_J @. @. \\
@. @VVV @VVV @. @. \\
0 @>>> J @>>> K @>>> L @>>> 0 \\
@. @VVV @VVV @| @. \\
0 @>>> M^1 @>>> X_0 @>>> L @>>> 0
\end{CD}$$

Since K is in $\operatorname{thick}(X)$, so too is X_0, and since $\operatorname{Ext}^1_R(X, J) = 0$ we have $M^1 \in X^\perp$. This yields the required five term exact sequence

$$0 \to M^1 \to X_0 \to M \to M^0 \to X_1 \to 0.$$

Next, the thick subcategory ${}^\perp(X^\perp)$ contains X, so contains all of $\operatorname{thick}(X)$. Conversely, given $M \in {}^\perp(X^\perp)$, the map $M \to M^0$ in the five term sequence must be zero, and then the short exact sequence $0 \to M^1 \to X^0 \to M \to 0$ must be split. Thus M is a summand of X^0, so lies in thick(X).

It now follows that $X^\perp = \text{thick}(X)^\perp$. So, writing L and N for the images of $X_0 \to M$ and $M \to M^0$ in the five term sequence, we have for all $X' \in \text{thick}(X)$ that $\text{Hom}_R(X', N) = 0$ and isomorphisms $\text{Hom}_R(X', X_0) \xrightarrow{\sim} \text{Hom}_R(X', L) \xrightarrow{\sim} \text{Hom}_R(X', M)$. Similarly for all $Y \in X^\perp$ we have $\text{Hom}_R(L, Y) = 0$ and isomorphisms $\text{Hom}_R(M^0, Y) \xrightarrow{\sim} \text{Hom}_R(N, Y) \xrightarrow{\sim} \text{Hom}_R(M, Y)$. □

We also have the dual construction, which we state for clarity.

Proposition 9.15. *Let X be a rigid module. Then every module M fits into a five term exact sequence*

$$0 \to X^1 \to M_0 \to M \to X^0 \to M_1 \to 0$$

with $X^0, X^1 \in \text{thick}(X)$ and $M_0, M_1 \in {}^\perp X$.

Moreover, $({}^\perp X)^\perp = \text{thick}(X)$ and we have natural isomorphisms

$$\text{Hom}_R(X^0, X') \xrightarrow{\sim} \text{Hom}_R(M, X') \quad \textit{for all } X' \in \text{thick}(X)$$

and

$$\text{Hom}_R(N, M_0) \xrightarrow{\sim} \text{Hom}_R(N, M) \quad \textit{for all } N \in {}^\perp X.$$

Let $\mathcal{T} \subset \text{mod}\, R$ be thick abelian. We define $K_0(\mathcal{T})$ to be the span inside $\Gamma := K_0(R)$ of all $[M]$ for $M \in \mathcal{T}$.

Corollary 9.16. *Let X be rigid. Then $\Gamma = K_0(\text{thick}(X)) \oplus K_0(X^\perp)$ and similarly $\Gamma = K_0({}^\perp X) \oplus K_0(\text{thick}(X))$.*

Proof. Using the two five term sequences we know that

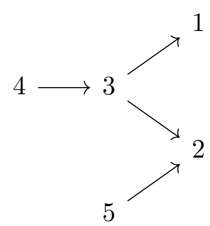
$$K_0(\text{thick}(X)) + K_0(X^\perp) = \Gamma = K_0(\text{thick}(X)) + K_0({}^\perp X).$$

Also, $\langle w, x\rangle = 0 = \langle x, y\rangle$ for all $w \in K_0({}^\perp X)$, $x \in K_0(\text{thick}(X))$ and $y \in K_0(X^\perp)$. Thus, given $z \in K_0(\text{thick}(X)) \cap K_0(X^\perp)$, we have $\langle w + x, z\rangle = 0$ for all $w \in K_0({}^\perp X)$ and $x \in K_0(\text{thick}(X))$, so $\langle -, z\rangle \equiv 0$. Since the bilinear form $\langle -, -\rangle$ is non-degenerate, we deduce that $z = 0$.

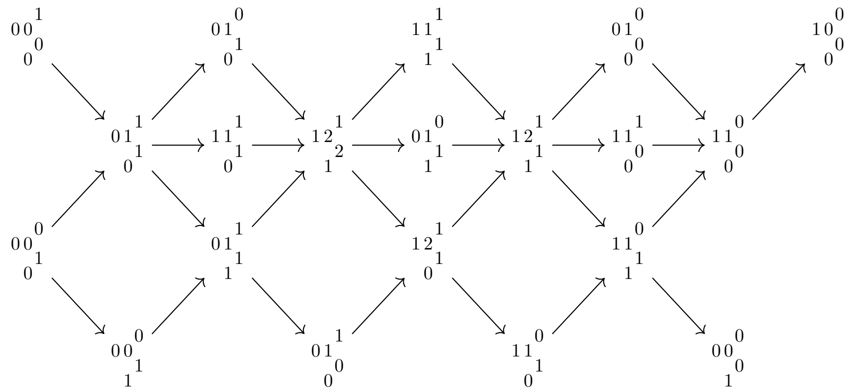
Thus $K_0(\text{thick}(X)) \cap K_0(X^\perp) = 0$, and similarly $K_0(\text{thick}(X)) \cap K_0({}^\perp X) = 0$. □

9.6 Examples

Consider the following quiver of type $\mathbb{D}5$.



We write elements in the Grothendieck group as arrays of numbers in the shape of the quiver. We can then draw the classes of the indecomposables as usual (the Auslander-Reiten quiver), projectives on the left, injectives on the right.

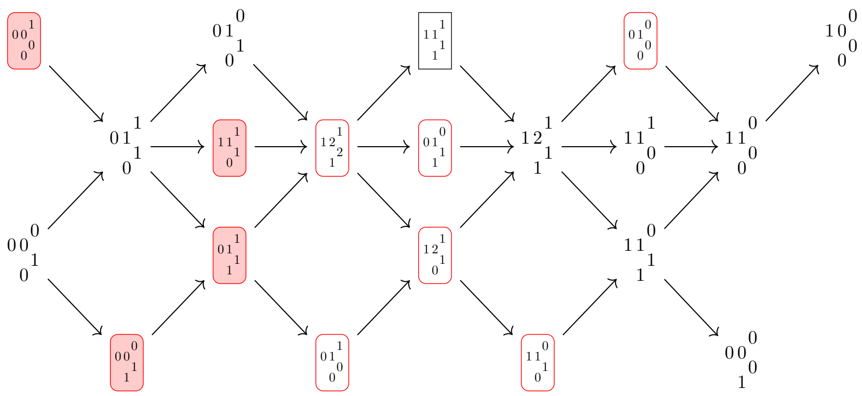


Given an exceptional module X, we will express $\langle X, - \rangle$ as an array of numbers in the shape of the quiver, so that $\langle X, Y \rangle$ can be computed by taking the sum of the products over the vertices. As X is indecomposable preprojective, we cannot have both $\operatorname{Hom}_R(X, Y)$ and $\operatorname{Ext}^1_R(X, Y) \cong D \operatorname{Hom}_R(Y, \tau X)$ non-zero. Thus $X^\perp$ consists of those indecomposables Y such that $\langle X, Y \rangle = 0$. (In general, this holds whenever X is indecomposable preprojective or postinjective.)

We will take one exceptional from each τ-orbit. We will mark X with a black square, and the indecomposables in $X^\perp$ with a red square; the relative projectives in $X^\perp$ will be shaded light red. Finally we will give the quiver Q such that $X^\perp$ is equivalent to $\operatorname{mod} kQ$.

Case 1

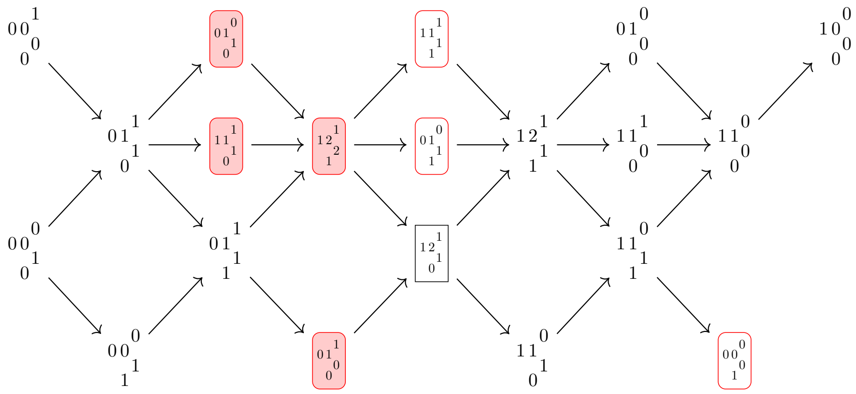
$X = \begin{array}{ccc} & & 1 \\ 1 & 1 & \\ & & 1 \\ & 1 & \end{array}$ and $\langle X, - \rangle = \begin{array}{ccc} & & 0 \\ 1 & 0 & \\ & & -1 \\ & 1 & \end{array}.$



Here Q is the quiver $\bullet \to \bullet \leftarrow \bullet \to \bullet$ of type $\mathbb{A}_4$.

Case 2

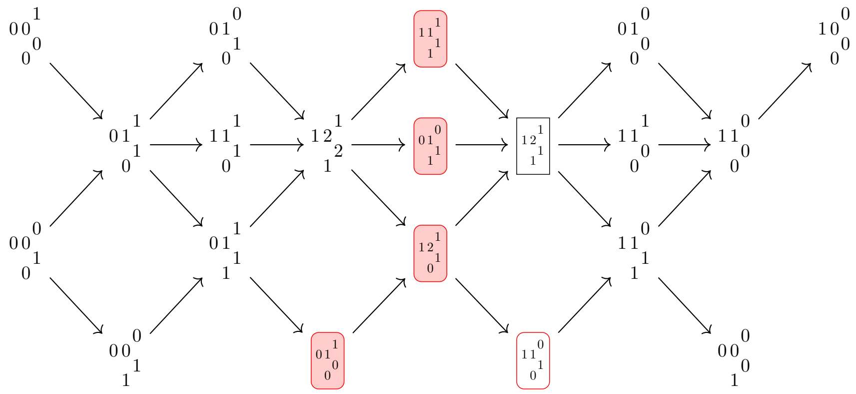
$X = \begin{smallmatrix} & & 1 \\ 1 & 2 & \\ & & 1 \\ & 0 & \end{smallmatrix}$ and $\langle X, - \rangle = \begin{smallmatrix} & & -1 \\ 1 & 1 & \\ & & -1 \\ & 0 & \end{smallmatrix}$.



Here Q is the quiver $\bullet \leftarrow \bullet \rightarrow \bullet \quad \bullet$ of type $\mathbb{A}_3 \sqcup \mathbb{A}_1$.

Case 3

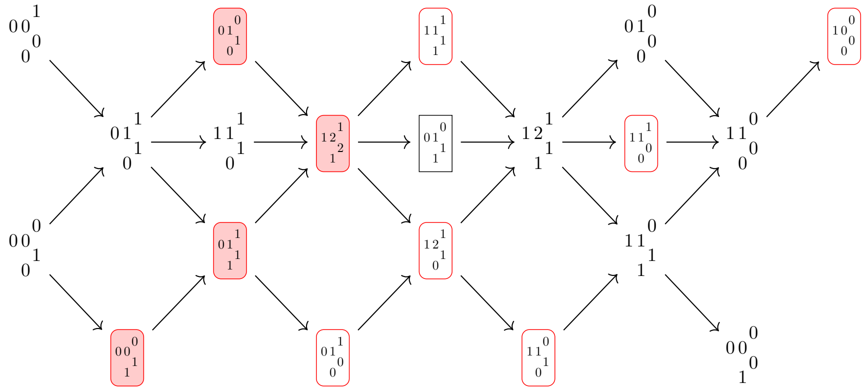
$X = \begin{smallmatrix} & & 1 \\ 1 & 2 & \\ & & 1 \\ & 1 & \end{smallmatrix}$ and $\langle X, - \rangle = \begin{smallmatrix} & & -1 \\ 1 & 1 & \\ & & -2 \\ & 1 & \end{smallmatrix}$.



Here Q is the quiver $\bullet \rightarrow \bullet \quad \bullet \quad \bullet$ of type $\mathbb{A}_2 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_1$.

Case 4

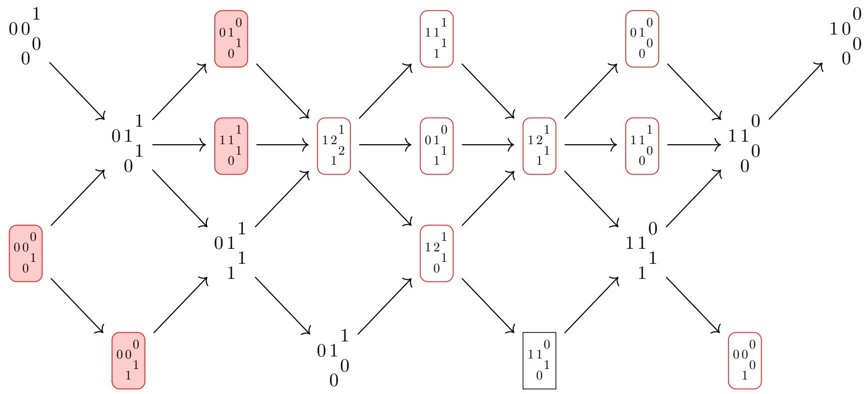
$X = \begin{smallmatrix} & & 0 \\ 0 & 1 & \\ & & 1 \\ & 1 & \end{smallmatrix}$ and $\langle X, - \rangle = \begin{smallmatrix} & & 0 \\ -1 & 0 & \\ & & 1 \\ & 0 & \end{smallmatrix}.$



Here Q is the quiver $\bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$ of type $\mathbb{A}_4$.

Case 5

$X = \begin{smallmatrix} & & 0 \\ 1 & 1 & \\ & & 1 \\ & 0 & \end{smallmatrix}$ and $\langle X, - \rangle = \begin{smallmatrix} & & -1 \\ 1 & 0 & \\ & & 0 \\ & 0 & \end{smallmatrix}.$



Here Q is the quiver $\begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \rightarrow & \bullet & \leftarrow & \bullet \end{array}$ of type $\mathbb{D}_4$.

9.7 Projective generators

Let $\mathcal{T} \subset \operatorname{mod} R$ be a thick abelian subcategory. A module $Y \in \mathcal{T}$ is a relative projective if $\operatorname{Ext}^1_R(Y,M) = 0$ for all $M \in \mathcal{T}$, and is a generator if there is an epimorphism $Y^r \twoheadrightarrow M$ for all $M \in \mathcal{T}$.

We will first show that if X is rigid, then the thick subcategories $X^\perp$, $\operatorname{thick}(X)$ and ${}^\perp X$ always contain a (relative) projective generator. We then show that if Y is a projective generator for a thick abelian subcategory $\mathcal{T}$, then we have an equivalence $\mathcal{T} \cong \operatorname{mod}\operatorname{End}_R(Y)$, and that $\operatorname{End}_R(Y)$ is again a finite dimensional hereditary algebra.

Lemma 9.17 (Bongartz complement)**.** *Let X be rigid. Then $X^\perp$,* $\operatorname{thick}(X)$ *and ${}^\perp X$ all contain a (relative) projective generator.*

Proof. Consider the five term exact sequence for R and set $Y := R^0 \in X^\perp$. Given $M \in X^\perp$, take an epimorphism $R^m \twoheadrightarrow M$. Since $\operatorname{Hom}_R(R,M) \cong \operatorname{Hom}_R(Y,M)$, this map comes from some $Y^m \to M$, necessarily surjective, so Y is a generator. Moreover, writing L and N for the images of $X_0 \to R$ and $R \to Y$, we have $\operatorname{Hom}_R(L,M) = 0 = \operatorname{Ext}^1_R(R,M)$, and so $\operatorname{Ext}^1_R(Y,M) \xrightarrow{\sim} \operatorname{Ext}^1_R(N,M) = 0$. Thus Y is relative projective.

For the left perpendicular category, the corresponding construction using the injective DR yields a (relative) injective cogenerator Y of ${}^\perp X$. In particular ${}^\perp X = \operatorname{thick}(Y)$ and $\operatorname{thick}(X) = Y^\perp$, so contains a (relative) projective generator as above.

We now repeat to get a (relative) injective cogenerator Z of ${}^\perp Y$, in which case ${}^\perp X = \operatorname{thick}(Y) = Z^\perp$, and so also has a (relative) projective generator. $\square$

In fact, when X has no injective summands, so $\tau\tau^- X \cong X$, we can use the Auslander-Reiten Formula to show that ${}^\perp X = (\tau^- X)^\perp$.

Lemma 9.18. *Let X be rigid, set $E := \operatorname{End}_R(X)$, and consider the adjoint pair of functors $F\colon \operatorname{mod} E \to \operatorname{mod} R$, $M \mapsto M \otimes_E X$, and $G\colon \operatorname{mod} R \to \operatorname{mod} E$, $N \mapsto \operatorname{Hom}_R(X,N)$. Then the natural transformation $\eta\colon \mathrm{id}_{\operatorname{mod} E} \to GF$ (the unit) is an isomorphism on all projective E-modules, and $\varepsilon\colon FG \to \mathrm{id}_{\operatorname{mod} R}$ (the counit) is an isomorphism on all $N \in \operatorname{add}(X)$.*

In other words, these functors restrict to an equivalence of additive categories $\operatorname{add}(E) \cong \operatorname{add}(X)$.

Proof. Recall that X is naturally an E-R-bimodule, so we do have the functors F and G. Moreover, $F(E) \cong X$ and $G(X) = E$, so they restrict to functors between $\operatorname{add}(E)$ and $\operatorname{add}(X)$.

Next, to say that (F,G) form an adjoint pair is to say that we have a natural isomorphism $\operatorname{Hom}_R(FM,N) \cong \operatorname{Hom}_E(M,GN)$ for all $M \in \operatorname{mod} E$ and $N \in \operatorname{mod} R$ (so a natural isomorphism of bifunctors).

In particular, the counit $\varepsilon_N\colon FGN \to N$ corresponds under the isomorphism $\operatorname{Hom}_E(FGN,N) \cong \operatorname{Hom}_R(GN,GN)$ to the identity on GN. More precisely, $\varepsilon_N\colon \operatorname{Hom}_R(X,N) \otimes_E X \to N$ sends $f \otimes x$ to $f(x)$. Thus ε_X is the isomorphism $E \otimes_E X \xrightarrow{\sim} X$, and hence by Lemma 7.5 we know that η_N is an isomorphism for all $N \in \operatorname{add}(X)$.

Similarly the unit $\eta_M\colon M \to GFM$ corresponds to the identity on FM. More precisely $\eta_M(m)$ is the map $X \to M \otimes_E X$, $x \mapsto m \otimes x$. Thus, given $\alpha \in E = \operatorname{End}_R(X)$, the composition $\varepsilon_X \eta_E(\alpha)$ is the map $X \to E \otimes_R X \xrightarrow{\sim} X$,

$x \mapsto \alpha(x)$, so equals α. Hence η_E is an isomorphism, so η_M is an isomorphism for all $M \in \mathrm{add}(E)$. □

When X is a relative projective generator in $\mathrm{thick}(X)$, then we can extend this to an equivalence $\mathrm{mod}\, E \cong \mathrm{thick}(X)$.

Proposition 9.19. *Let R be a finite dimensional hereditary algebra, X a rigid module, and $E = \mathrm{End}_R(X)$. If X is a relative projective generator in $\mathrm{thick}(X)$, then we have an equivalence $\mathrm{mod}\, E \cong \mathrm{thick}(X)$. Moreover, E is again a finite dimensional hereditary algebra.*

Proof. Take $M \in \mathrm{mod}\, E$, and take a presentation $E^r \to E^s \to M \to 0$. As in the proof of Lemma 7.13, it follows that the functor $F = - \otimes_E X$ is right exact, so we obtain an exact sequence $X^r \to X^s \to FM \to 0$ in $\mathrm{thick}(X)$. Then, as X is a relative projective, the functor G is exact on $\mathrm{thick}(X)$, and so we obtain an exact commutative diagram

$$\begin{array}{ccccccc} E^r & \longrightarrow & E^s & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle \eta_{E^r}}\downarrow\wr & & {\scriptstyle \eta_{E^s}}\downarrow\wr & & \downarrow{\scriptstyle \eta_M} & & \\ GF(E^r) & \longrightarrow & GF(E^s) & \longrightarrow & GF(M) & \longrightarrow & 0 \end{array}$$

Thus η_M is an isomorphism by the Snake Lemma.

Conversely, take $N \in \mathrm{thick}(X)$. Since X is a generator there is an exact sequence $X^r \to X^s \to N \to 0$ in $\mathrm{thick}(X)$. Again, G is exact and F is right exact, so we obtain the exact commutative diagram

$$\begin{array}{ccccccc} FG(X^r) & \longrightarrow & FG(X^s) & \longrightarrow & N & \longrightarrow & 0 \\ {\scriptstyle \varepsilon_{X^r}}\downarrow\wr & & {\scriptstyle \varepsilon_{X^s}}\downarrow\wr & & \downarrow{\scriptstyle \varepsilon_N} & & \\ X^r & \longrightarrow & X^s & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Thus ε_N is an isomorphism by the Snake Lemma.

Finally, we will use Proposition 6.1 (6) to prove that E is hereditary. Given $f\colon M \to M'$ in $\mathrm{mod}\, E$, we apply F and use that R is hereditary to get a short exact sequence $0 \to F(M) \to \mathrm{Im}(F(f)) \oplus N \to F(M') \to 0$. This lies in $\mathrm{thick}(X)$, so we can apply G to obtain the short exact sequence $0 \to M \to \mathrm{Im}(f) \oplus G(N) \to M' \to 0$ in $\mathrm{mod}\, E$. Hence E is hereditary. □

Corollary 9.20. *Let X be exceptional. Then $\mathrm{thick}(X) = \mathrm{add}(X)$.*

Proof. Since X is exceptional we know that $E = \mathrm{End}_R(X)$ is a division algebra, and hence (as for fields) that $\mathrm{mod}\, E = \mathrm{add}(E)$. Under the equivalence $\mathrm{add}(E) \cong \mathrm{add}(X)$ we conclude that $\mathrm{add}(X) = \mathrm{thick}(X)$. □

9.8 The tubular type of a tame hereditary algebra

Let R be a finite dimensional, indecomposable, tame hereditary algebra of rank n. Recall that we have representatives S_x with $x \in \mathbb{X}$ for the τ-orbits of regular-simple modules, that S_x has finite order p_x under τ, and that $\sum_{\mathbb{X}}(p_x - 1) \leq n-2$. The list of numbers $p_x > 1$ is called the tubular type of R.

We also know that R is homogeneous, that is $p_x = 1$ for all x, if and only if $n = 2$.

Theorem 9.21. *Let R be a finite dimensional, indecomposable, tame hereditary algebra of tubular type $(p_1,\dots,p_r)$. Take regular-simples S_i from each of the corresponding orbits. Then $X := \bigoplus_i S_i$ is rigid, and $X^\perp \cong \operatorname{mod} E$ for some finite dimensional tame hereditary algebra E of tubular type $(p_1-1,\dots,p_r-1)$. Moreover, the τ_E-orbits of regular-simples are again indexed by $\mathbb{X}$.*

Proof. We know that each S_i is exceptional, and there are no homomorphisms or extensions between distinct orbits, so $X := \bigoplus_i S_i$ is rigid. Using the theory developed in the previous section we have $X^\perp \cong \operatorname{mod} E$ where E is again a finite dimensional hereditary algebra.

We know that $\operatorname{Ext}^1_R(S_i,\tau S_i) \cong D\operatorname{End}_R(S_i)$, so up to equivalence there exists a unique non-split short exact sequence $0 \to \tau S_i \xrightarrow{\iota} S_i^{[2]} \xrightarrow{\pi} S_i \to 0$. Then $S_i^{[2]}$ lies in $X^\perp$. Moreover, by considering dimensions, $\operatorname{Ext}^1_R(S_i,\tau S_i)$ must be a simple right $\operatorname{End}_R(S_i)$-module. Applying $\operatorname{Hom}_R(S_i,-)$, the connecting map $\operatorname{End}_R(S_i) \to \operatorname{Ext}^1_R(S_i,\tau S_i)$ is a non-zero map between simple $\operatorname{End}_R(S_i)$-modules, hence an isomorphism. Since $\operatorname{Hom}_R(S_i,\tau S_i) = 0 = \operatorname{Ext}^1_R(S_i,S_i)$ it follows that $S_i^{[2]} \in S_i^\perp$, and hence also lies in $X^\perp$.

If $p_i > 2$, then also $\tau^j S_i \in S_i^\perp$ for $2 \le j < p_i$, and hence they also lie in $X^\perp$. It follows that

$$m_i\delta = \sum_{0\le j<p_i} [\tau^j S_i] = [S_i^{[2]}] + \sum_{2\le j<p_i} [\tau^j S_i] \in K_0(X^\perp).$$

Now the Euler form on $\operatorname{mod} E$ is just the restriction of the Euler form on $\operatorname{mod} R$ to the sublattice $K_0(X^\perp)$, and this contains a multiple of δ. Thus the Grothendieck group $K_0(E)$ is of affine type, so E is tame, and for $M \in X^\perp$, its defect as an E-module is (a multiple of) its defect as an R-module. In particular, the regular E-modules are precisely the modules $M \in X^\perp$ which are regular as an R-module.

Thus, if S_x is regular-simple of order $p_x = 1$, then $S_x \in X^\perp$ and S_x is a regular-simple E-module. Similarly, if $p_i > 2$, then each $\tau^j S_i$ for $2 \le j < p_i$ is regular-simple as an E-module.

Finally, $S_i^{[2]}$ is regular-simple in $X^\perp$. For, given a regular submodule T of $S_i^{[2]}$, we obtain an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & T & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau S_i & \xrightarrow{\iota} & S_i^{[2]} & \xrightarrow{\pi} & S_i & \longrightarrow & 0
\end{array}$$

Since τS_i and S_i are both regular-simples, K is either 0 or all of τS_i, and I is either zero or all of S_i. Thus T is either 0, τS_i, S_i or $S_i^{[2]}$. Moreover, S_i cannot occur since $S_i^{[2]} \in S_i^\perp$, and $\tau S_i \notin X^\perp$, proving that T is regular-simple in $X^\perp$ as claimed. (In fact, we have an algebra isomorphism $\operatorname{End}_R(S_i^{[2]}) \cong \operatorname{End}_R(S_i)$.)

We claim that this gives all regular-simple modules in $\operatorname{mod} E$. Let $0 \neq S \in X^\perp$ be regular. Then it has a regular-simple submodule as an R-module, so either S_x with $p_x = 1$, or else $\tau^j S_i$ with $2 \le j < p_i$, or else S_i or τS_i. In the first two cases these are again regular modules in $X^\perp$, and the third cannot occur since $S \in X^\perp$. In the fourth case we apply $\operatorname{Hom}_R(-,S)$ to the exact sequence defining $S_i^{[2]}$ to get $\operatorname{Hom}_R(S_i^{[2]},S) \cong \operatorname{Hom}_R(\tau S_i,S)$, so there is a non-zero homomorphism $S_i^{[2]} \to S$, and this is necessarily injective since $S_i^{[2]}$ is regular-simple as an E-module. Thus there are no other regular-simples E-modules.

Although there is no nice way to construct the Auslander-Reiten translate of $\operatorname{mod} E$ in terms of that on $\operatorname{mod} R$, we can compute its orbits on the regular-simples using that $\operatorname{Ext}^1_E(S, S') \neq 0$ if and only if $S' \cong \tau_E S$. We have $\operatorname{Ext}^1_R(S_x, S_x) \neq 0$ for all regular-simples S_x with $p_x = 1$. Similarly $\operatorname{Ext}^1_R(\tau^j S_i, \tau^{j+1} S_i)$ for all $2 \leq j < p_i - 1$. Also, using the short exact sequence for $S_i^{[2]}$ we have $\operatorname{Ext}^1_R(S_i^{[2]}, \tau^2 S_i) \twoheadrightarrow \operatorname{Ext}^1_R(\tau S_i, \tau^2 S_i) \neq 0$, and similarly $\operatorname{Ext}^1_R(\tau^- S_i, S_i^{[2]}) \twoheadrightarrow \operatorname{Ext}^1_R(\tau^- S_i, S_i) \neq 0$.

For $p_i > 2$ this gives the τ_E-orbit of regular simples $S_i^{[2]}, \tau^2 S_i, \ldots, \tau^{p_i - 1} S_i$, of order $p_i - 1$. For $p_i = 2$ this gives $\operatorname{Ext}^1_R(\tau S_i, S_i^{[2]}) \neq 0$, so $\operatorname{Ext}^1_R(S_i^{[2]}, S_i^{[2]}) \neq 0$ and $S_i^{[2]}$ has period $1 = p_i - 1$. This proves that E has tubular type $(p_1 - 1, \ldots, p_r - 1)$, and that the τ_E-orbits of regular simples are again indexed by $\mathbb{X}$. □

Corollary 9.22. *Let R be an indecomposable, finite dimensional tame hereditary algebra. Let $\mathbb{X}$ index the τ-orbits of regular-simple R-modules. Let S_i be one regular-simple from each τ-orbit of size $p_i > 1$, and set $\mathcal{X} := \{\tau^j S_i : 0 \leq j < p_i - 1\}$. Then $\mathcal{X}^\perp \cong \operatorname{mod} E$ for a finite dimensional tame hereditary algebra of rank two. Moreover, the corresponding embedding $\operatorname{mod} E \to \operatorname{mod} R$ identifies the regular-simple E-module with the indecomposable regular R-modules $S_x^{[p_x]}$ having regular composition factors (from top to bottom) $S_x, \tau S_x, \ldots, \tau^{p_x - 1} S_x$.*

Proof. Let R have tubular type $(p_1, \ldots, p_r)$ and set $X := \bigoplus_i S_i$. By the theorem $X^\perp \cong \operatorname{mod} E'$, where E' has tubular type $(p_1 - 1, \ldots, p_r - 1)$. The corresponding embedding $\operatorname{mod} E' \to \operatorname{mod} R$ identifies the regular-simple E'-modules with the regular R-modules S_x for $p_x = 1$, $S_i^{[2]}$ and $\tau^j S_i$ for $2 \leq j < p_i - 1$.

We can now repeat, using the regular-simple E'-modules $S_i^{[2]}$ for $p_i > 2$. This gives an embedding $\operatorname{mod} E'' \to \operatorname{mod} R$, where E'' has tubular type $(p_i - 2)$ (involving just those p_i with $p_i > 2$), whose image is right perpendicular to all S_i for $p_i > 1$ and all $S_i^{[2]}$ for $p_i > 2$. Note that this is the same as being perpendicular to all S_i for $p_i > 1$ and all τS_i for $p_i > 2$. Moreover, the regular-simple E''-modules are identified with S_x for $p_x = 1$, $S_i^{[2]}$ for $p_i = 2$, and $S_i^{[3]}$ together with $\tau^j S_i$ with $3 \leq j < p_i - 1$ for $p_i > 2$. Here, $S_i^{[3]}$ is the middle term of a non-split extension of $S_i^{[2]}$ by $\tau^2 S_i$, so has regular composition factors (from top to bottom) $S_i, \tau S_i, \tau^2 S_i$.

Continuing in this way we see that there is a tame homogeneous algebra E, necessarily of rank two, and an embedding $\operatorname{mod} E \to \operatorname{mod} R$. The image is identified with $\mathcal{X}^\perp$, where $\mathcal{X} = \{\tau^j S_i : 0 \leq j < p_i - 1\}$. The regular-simple E-modules are identified with the indecomposable regular modules $S_x^{[p_x]}$, having regular composition factors (from top to bottom) $S_x, \tau S_x, \ldots, \tau^{p_x - 1} S_x$ for all $x \in \mathbb{X}$. □

Corollary 9.23. *Let R be an indecomposable, finite dimensional, tame hereditary algebra of rank n and tubular type $(p_1, \ldots, p_r)$. Then the Grothendieck group has basis δ, $[\tau^j S_i]$ for $0 \leq j < p_i - 1$ and $[P]$, where P is indecomposable preprojective of defect -1. In particular, $\sum_{\mathbb{X}} (p_x - 1) = n - 2$.*

Proof. Set $\mathcal{X} = \{\tau^j S_i : 0 \le j < p_i - 1\}$. By Theorem 9.4 we know that $K_0(\mathrm{thick}(\mathcal{X}))$ has basis $[\tau^j S_i]$, so rank $\sum_i (p_i - 1)$. On the other hand, using induction as in the proof of the corollary above, together with the earlier result from Corollary 9.16, we conclude that $\Gamma = K_0(\mathrm{thick}(\mathcal{X})) \oplus K_0(\mathcal{X}^\perp)$, and that $K_0(\mathcal{X}^\perp)$ has rank two. Moreover, some positive multiple of δ lies in $K_0(\mathcal{X}^\perp)$, so necessarily $\delta \in K_0(\mathcal{X}^\perp)$. Finally, if $P \in \mathcal{X}^\perp$ is a non-zero relative projective of maximal defect, then necessarily δ and $[P]$ form a basis for $K_0(\mathcal{X}^\perp)$, and so P has defect -1. □

Corollary 9.24. *Let R be a finite dimensional, indecomposable, tame hereditary algebra of tubular type $(p_1, \dots, p_r)$. Then the action of the Coxeter element c on the Grothendieck group Γ has characteristic polynomial*

$$\chi_c(t) = (t-1)^2 \prod_i \frac{t^{p_i} - 1}{t - 1}.$$

It follows that the tubular type of R depends only on the conjugacy class of the Coxeter transformation. In particular, if the Dynkin diagram is a tree, then all Coxeter elements are conjugate and so the tubular type depends only on the Dynkin type.

Proof. We compute the action of the Coxeter transformation with respect to the basis δ, $[\tau^j S_i] = c^j[S_i]$ for $0 \le j < p_i - 1$, and $[P]$, where P is an indecomposable preprojective of defect -1.

We know that c fixes δ, and sends $c^j[S_i]$ to $c^{j+1}[S_i]$ for $0 \le j < p_i - 1$. Since $\sum_{0 \le j < p_i} [\tau^j S_i] = m_i \delta$ for some $m_i > 0$, we see that c sends $[\tau^{p_i - 2} S_i]$ to $m_i \delta - \sum_{0 \le j < p_i - 1} [\tau^j S_i]$. Finally, $c[P] - [P]$ has defect zero, so lies in the span of δ and the $[\tau^j S_i]$. Thus the matrix for this action has the following form

$$\begin{pmatrix} 1 & & & & & \\ v_1 & M_1 & & & & \\ v_2 & & M_2 & & & \\ \vdots & & & \ddots & & \\ v_r & & & & M_r & \\ * & * & * & \cdots & * & 1 \end{pmatrix}, \quad \text{where } M_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -1 & \cdots & -1 & -1 \end{pmatrix}$$

the matrix M_i has size $p_i - 1$, and $v_i = (0, \dots, 0, m_i)^t$. Thus $\chi_c(t) = (t - 1)^2 \prod_i \det(t - M_i)$, and by expanding down the first column and using induction we see that $\det(t - M_i) = t^{p_i - 1} + \cdots + t + 1 = (t^{p_i} - 1)/(t - 1)$. □

The following table lists the Dynkin diagrams of affine type, together with a choice of Coxeter element $c = s_n \cdots s_1$, indicated by the labelling of the vertices; the defect ∂, indicated by the dot product with an element in the Grothendieck group; the class of one regular-simple from each τ-orbit of order $p > 1$, together with its period p.

Dynkin type	Coxeter element	defect	regular simple	period
$\widetilde{\mathbb{A}}_{n-1}$	2—3⋯⋯p 1 n p+1—p+2⋯n−1	0 0 0 −1 1 0 0 0	1 0 0 0 0 0 0 0	p
			0 0 0 0 0 1 0 0	$n-p$
$\widetilde{\mathbb{D}}_{n-1}$	1 n−1 3—4⋯n−2 2 n	−1 1 0 0 0 −1 1	1 1 1 1 1 0 0	2
			1 0 1 1 1 0 1	2
			0 0 0 0 1 0 0	$n-3$
$\widetilde{\mathbb{E}}_6$	7 | 6 | 3–2–1–4–5	1 1 1 1 −3 1 1	0 1 0 1 1 1 0	2
			0 0 0 1 1 1 1	3
			0 0 1 1 1 1 0	3
$\widetilde{\mathbb{E}}_7$	8 4–3–2–1–5–6–7	2 1 1 1 −4 1 1 1	1 0 1 1 2 2 1 1	2
			1 0 0 1 1 1 0 0	3
			0 0 0 1 1 1 1 1	4
$\widetilde{\mathbb{E}}_8$	9 6–5–4–3–2–1–7–8	3 1 1 1 1 1 −6 2 2	2 0 1 1 2 2 3 2 1	2
			1 0 0 1 1 1 2 2 1	3
			1 0 0 0 0 1 1 1 0	5

Dynkin type	Coxeter element	defect	regular simple	period
$\widetilde{\mathbb{A}}'_{n-1}$	$1\overset{(1,2)}{\text{—}}2\cdots n{-}1\overset{(1,2)}{\text{—}}n$	$-2\,0\,0\,1$	$0\,1\,0\,0$	$n-1$
$\widetilde{\mathbb{B}}_{n-1}$	$\begin{smallmatrix}1 & \\ & 3\cdots n{-}1\overset{(1,2)}{\text{—}}n \\ 2 & \end{smallmatrix}$	$\begin{smallmatrix}-1\\-1\end{smallmatrix}\,0\;0\;1$	$\begin{smallmatrix}1\\0\end{smallmatrix}\,1\;1\;1$ $\begin{smallmatrix}0\\0\end{smallmatrix}\,1\;0\;0$	2 $n-2$
$\widetilde{\mathbb{B}}^t_{n-1}$	$\begin{smallmatrix}1 & \\ & 3\cdots n{-}1\overset{(2,1)}{\text{—}}n \\ 2 & \end{smallmatrix}$	$\begin{smallmatrix}-1\\-1\end{smallmatrix}\,0\;0\;2$	$\begin{smallmatrix}2\\0\end{smallmatrix}\,2\;2\;1$ $\begin{smallmatrix}0\\0\end{smallmatrix}\,1\;0\;0$	2 $n-2$
$\widetilde{\mathbb{C}}_{n-1}$	$1\overset{(1,2)}{\text{—}}2\cdots n{-}1\overset{(2,1)}{\text{—}}n$	$-1\,0\,0\,1$	$0\,1\,0\,0$	$n-1$
$\widetilde{\mathbb{C}}^t_{n-1}$	$1\overset{(2,1)}{\text{—}}2\cdots n{-}1\overset{(1,2)}{\text{—}}n$	$-1\,0\,0\,1$	$0\,1\,0\,0$	$n-1$
$\widetilde{\mathbb{F}}_4$	$1\text{—}2\text{—}3\overset{(1,2)}{\text{—}}4\text{—}5$	$-1\,-1\,-1\,1\,1$	$0\,1\,1\,2\,0$ $0\,0\,1\,1\,0$	2 3
$\widetilde{\mathbb{F}}^t_4$	$1\text{—}2\text{—}3\overset{(2,1)}{\text{—}}4\text{—}5$	$-1\,-1\,-1\,2\,2$	$0\,1\,1\,1\,0$ $0\,0\,2\,1\,0$	2 3
$\widetilde{\mathbb{G}}_2$	$1\text{—}2\overset{(1,3)}{\text{—}}3$	$-1\,-1\,1$	$0\,1\,1$	2
$\widetilde{\mathbb{G}}^t_2$	$1\text{—}2\overset{(3,1)}{\text{—}}3$	$\partial = -1\,-1\,3$	$0\,3\,1$	2