Representations of hereditary algebras
Exercises 1

1. Let $0 \to U \xrightarrow{f} M \xrightarrow{g} V \to 0$ be a short exact sequence of R-modules. Show that M has finite length if and only if both U and V have finite length, in which case the simple subquotientes of M are precisely those of U and V combined (with multiplicities). In particular, $\ell(M) = \ell(U) + \ell(V)$.

   > Hint: Given composition series for U and V, show how to construct a composition series for M, and identify the simple subquotients. For the converse it is now enough to show that M finite length implies both U and V finite length. So, starting from a composition series for M, show that the term-wise intersection with U gives a chain of submodules with subquotients either simple or zero. Similarly for the term-wise images in V.

2. Let M_i be a family of R-modules indexed by a set I. Prove the following about their direct sum $\coprod_i M_i$ and direct product $\prod_i M_i$.

   (a) For all X the map
   $$\mathrm{Hom}_R(\coprod_i M_i, X) \to \prod_i (M_i, X), \quad f \mapsto (f\iota_i),$$
   is a bijection (in fact an isomorphism of left $\mathrm{End}_R(X)$-modules).

   (b) For all X the map
   $$\mathrm{Hom}_R(X, \prod_i M_i) \to \prod_i (X, M_i), \quad f \mapsto (\pi_i f),$$
   is a bijection (in fact an isomorphism of right $\mathrm{End}_R(X)$-modules).

3. Recall that a kernel for $f\colon L \to M$ is a map $i\colon K \to L$ such that $fi = 0$ and if $\alpha\colon X \to L$ satisfies $f\alpha = 0$, then there exists a unique $\bar{\alpha}\colon X \to K$ with $\alpha = i\bar{\alpha}$.

   Show that kernels are unique up to unique isomorphism; that is, if $i'\colon K' \to L$ is another kernel for f, then there exists a unique map $h\colon K \to K'$ with $i'h = i$, and moreover h is an isomorphism.

   > Hint: Use that i is a kernel to construct a map $h'\colon K' \to K$, and use uniqueness to prove that h and h' are inverses of one another.

4. Let M be an R-module. We say that a subset $S \subset M$ generates M provided every element of M can be written as a finite sum $\sum_i m_i r_i$ with $r_i \in R$ and $m_i \in S$. If we can take S to be finite, then we say that M is finitely generated. We say that M is Noetherian provided every submodule is finitely generated.

   (a) Let $0 \to U \xrightarrow{f} M \xrightarrow{g} V \to 0$ be a short exact sequence of R-modules. Show that M is Noetherian if and only if both U and V are Noetherian.

   > Hint: Suppose U and V are both Noetherian, and that $N \leq M$ is an arbitrary submodule. Take finite subsets $S, T \subset M$ such that S generates $N \cap U$ and the images $g(T)$ generate $g(N) \leq V$. Prove that the union $S \cup T$ generates N, and hence that N is finitely generated.

   (b) Recall that R is a (right) Noetherian ring provided it satisfies the ascending chain condition on right ideals. Explain why this is the same as saying that the (right) regular module R_R is Noetherian.

   Now let R be Noetherian. Prove that an R-module M is Noetherian if and only if it is finitely generated, which is if and only if it is a quotient of some R^n.

   In particular, it follows that every finitely generated R-module is the cokernel of some map $R^m \to R^n$, which we can represent as a matrix in $\mathbb{M}_{n\times m}(R)$. This was used in the classification of finitely generated modules over a principal ideal domain.

5. Let R be a principal ideal domain, and $p \in R$ a prime. Prove that R/pR is a simple R-module, and that R/p^nR has a composition series of length n with all simple subquotients isomorphic to R/pR.