Representations of hereditary algebras
Exercises 2

1. Recall the six term exact sequence in the statement of the Snake Lemma. Describe the map $\mathrm{Coker}(\lambda) \to \mathrm{Coker}(\mu)$, and prove exactness at $\mathrm{Coker}(\lambda)$.

2. Prove the following slightly stronger form of Lemma 4.6. Consider a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} \\ L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' \end{array}$$

where the left hand square is a push-out. Prove that the right hand square is a push-out if and only if the outer square is a push-out

$$\begin{array}{ccc} L & \xrightarrow{gf} & N \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\nu} \\ L' & \xrightarrow{g'f'} & N' \end{array}$$

3. Prove that the push-out yields a map on extension classes. Explicitly, suppose we have two push-out squares

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} \\ L' & \xrightarrow{f'} & M' \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\bar f} & \bar M \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\bar\mu} \\ L' & \xrightarrow{\bar f'} & \bar M' \end{array}$$

Given $\alpha\colon M \to \bar M$ such that $\alpha f = \bar f$, prove that there exists (a unique) $\alpha'\colon M' \to \bar M'$ such that $\alpha' f' = \bar f'$ and $\alpha'\mu = \bar\mu\alpha$.

Furthermore, if α is an isomorphism, so too is α'.

> Hint: let β be an inverse for α. Then $\beta\bar f = f$, so as above we obtain (a unique) $\beta'\colon \bar M' \to M'$ such that $\beta'\bar f' = f'$ and $\beta'\bar\mu = \mu\beta$. Now use uniqueness to prove that β' is an inverse for α'.

4. Suppose we have an exact commutative diagram

$$\begin{array}{ccccccccccc} \varepsilon\colon & & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ \varepsilon'\colon & & 0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0 \end{array}$$

Then $\lambda\varepsilon = \varepsilon'\nu$.

Hint: Take the push-out E of f and λ to obtain a commutative diagram with exact rows, where $\phi\theta = \mu$

$$\begin{array}{rccccccccc}
\varepsilon\colon & 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\theta} & & \| & & \\
\lambda\varepsilon\colon & 0 & \longrightarrow & L' & \xrightarrow{a} & E & \xrightarrow{b} & N & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\nu} & & \\
\varepsilon'\colon & 0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0
\end{array}$$

Now take the pull-back E' of g' and ν to obtain a commutative diagram with exact rows, where $\theta' e = \phi$

$$\begin{array}{rccccccccc}
\lambda\varepsilon\colon & 0 & \longrightarrow & L' & \xrightarrow{a} & E & \xrightarrow{b} & N & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle e} & & \| & & \\
\varepsilon'\nu\colon & 0 & \longrightarrow & L' & \xrightarrow{a'} & E' & \xrightarrow{b'} & N & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\nu} & & \\
\varepsilon'\colon & 0 & \longrightarrow & L' & \xrightarrow{f'} & M' & \xrightarrow{g'} & N' & \longrightarrow & 0
\end{array}$$

Deduce that e is an isomorphism.

Note that we used this result several times, in the proofs of Lemma 5.2, Lemma 5.9, Corollary 5.10, and Lemma 5.11. Also, this result implies the second half of Lemma 4.7 (and dually Lemma 4.10) which we did not prove in the lectures.

5. Prove Example 5.8. Let k be a field, R a k-algebra, and $D = \mathrm{Hom}_k(-, k)$ the usual vector space duality. Prove that if P is a projective left R-module, then $D(P)$ is an injective right R-module.

Hint: You will need the evaluation map $\mathrm{ev}\colon M \to D^2(M)$, where $\mathrm{ev}_m(f) := f(m)$ for all $m \in M$ and $f \in D(M)$.

Is it true that I an injective left R-module implies $D(I)$ a projective right R-module?