Representations of hereditary algebras
Exercises 3

Let L/K be a field extension, and consider L as a K-L-bimodule. Then we have the hereditary algebra

$$R = T_{L\otimes K}(L) \cong \begin{pmatrix} L & 0 \\ L & K \end{pmatrix}.$$

We wish to investigate indecomposable R-modules in some easy cases. We write $n := [L:K] = \dim_K L$ for the degree of the field extension.

Now, modules over R correspond to representations $(V_L, U_K; \theta)$, where V_L is an L-vector space, U_K is a K-vector space, and $V_L \xleftarrow{\theta} U_K \otimes_K L$ is an L-linear map. Choosing bases, we see that every such representation gives rise to a matrix $\theta \in \mathbb{M}_{e\times d}(L)$, where $e = \dim_L V$ and $d = \dim_K U$.

Next, homomorphisms $(V, U; \theta) \to (V', U'; \theta')$ are given by pairs (g, f), where $g\colon V \to V'$ is L-linear, $f\colon U \to U'$ is K-linear, and $g\theta = \theta'(f \otimes 1)$. Again, choosing bases, we see that two matrices $\theta, \theta' \in \mathbb{M}_{e\times d}(L)$ correspond to isomorphic representations if and only if there exist invertible matrices $g \in \mathrm{GL}_e(L)$ and $f \in \mathrm{GL}_d(K)$ such that $g\theta = \theta' f$.

Thus, for fixed integers (e, d) we wish to understand matrices in $\mathbb{M}_{e\times d}(L)$, but where we can apply L-linear row operations (the action of $\mathrm{GL}_e(L)$ on the left) and K-linear column operations (the action of $\mathrm{GL}_d(K)$ on the right). So classifying representations up to isomorphism can be rephrased as finding a normal form for such matrices under these actions.

In particular, the direct sum of two representations corresponds to a diagonal block matrix, so we should be able to read off the indecomposable summands from our normal form.

1. Suppose $n = 1$, so $L = K$. We are therefore studying matrices in $\mathbb{M}_{e\times d}(K)$ together with the usual row and column operations. In this case there are only three indecomposable representations; describe them.

2. Suppose $n = 2$, and let L have K-basis $1, x$.

   We may first put θ into row reduced form. Thus each row is now a vector of the form $(0, 0, 1, a + bx, 0, c + dx, e + fx)$, where the left-most non-zero entry is a 1. Next, using K-linear column operations, we can ensure every other entry is a multiple of x.

   We now consider the left-most column containing a non-zero multiple of x, say in column j. (Note that every element in this column is a multiple of x.) Rescaling the column, we may assume that the lowest non-zero entry in this column is precisely x, say in row i. Then by using K-linear row operations, we can make every other entry in this column zero, at the expense of introducing scalars from K in column i and rows above i coming from the pivot element. We can remove these, however by applying K-linear column operations, using the pivots from the rows above i. Finally, by applying K-linear column operations, we can now remove all other entries in row i.

For example, for $a, b, c \in K$, we have

$$\begin{pmatrix} 1 & 0 & 0 & ax & * \\ & 1 & 0 & bx & * \\ & & 1 & x & cx \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -a & 0 & * \\ & 1 & -b & 0 & * \\ & & 1 & x & cx \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & * \\ & 1 & 0 & 0 & * \\ & & 1 & x & 0 \end{pmatrix}$$

where $a, b \in K$.

Continuing in this way, we may assume that the matrix is now in row reduced form, and that each column contains at most one non-zero entry, which is either a 1 or an x. Finally, by rearranging the columns, we can write our matrix as a direct sum of matrices of the form $(1, x)$ or (1), or the trivial matrices of size 0×1 and 1×0.

In other words, we deduce that every representation is isomorphic to a direct sum of copies of the four representations

$$L \leftarrow 0, \quad L \xleftarrow{1} K, \quad L \xleftarrow{(1,x)} K^2, \quad 0 \leftarrow K.$$

Note that the first and last are simple, the first two are projective, and the last two are injective. Also, the second corresponds to the canonical embedding $L \hookleftarrow K$, whereas the third corresponds to the identification $L \xleftarrow{\sim} K^2$. Their endomorphism algebras are, respectively, L, K, L, K. Finally, their images in the Grothendieck group $\mathbb{Z}^2$, with basis $e_1 = [L \leftarrow 0]$ and $e_2 = [0 \leftarrow K]$ are, respectively,

$$(1, 0), \quad (1, 1), \quad (1, 2), \quad (0, 1).$$

3. Now suppose that $n = 3$.

   (a) Let $U, V \subset L$ be two K-vector subspaces of dimension two. Prove that there exists $\lambda \in L$ such that $\lambda U = V$.

   Hint. Assume first that U has K-basis $1, a$ and V has K-basis $1, b$. Then $U = V$ if and only if $1, a, b$ are linearly dependent. Otherwise, $1, a, b$ form a K-basis for L, so we can write $ab = p + qa + rb$ for some $p, q, r \in K$. Now find λ such that $\lambda U = V$. What about the general case?

   Deduce from this that if C and C' are K-vector space complements of K in L, then the representations $L \hookleftarrow C$ and $L \hookleftarrow C'$ are isomorphic.

   (b) Let $1, a, b$ be a K-basis for L, and let $M_{a,b}$ be the representation $L^2 \xleftarrow{\theta} K^3$ where $\theta = \left(\begin{smallmatrix} 1 & 0 & b \\ 0 & 1 & a \end{smallmatrix}\right)$. Show that all such representations are isomorphic.

   Hint. As above we can change the second row to $(0, 1, d)$, at the cost of making the first row $(1, x, y)$ for some $x, y \in L$. Subtract a suitable amount of the second row to get $(1, 0, c')$. Now explain how to finish the proof.

   We can characterise this as $L^2 \hookleftarrow U$ for some K-vector subspace U of L^2 of dimension three, which is *not* an L-vector subspace.

   (c) In fact, these two representations, together with the two simples $L \leftarrow 0$ and $0 \leftarrow K$, the non-simple projective $L \xleftarrow{1} K$ and the non-simple injective $L \overset{\sim}{\hookleftarrow} K^3$, are the only indecomposables up to isomorphism. Thus we have the six indecomposables

$$L \leftarrow 0, \quad L \hookleftarrow K, \quad L^2 \hookleftarrow U \quad L \hookleftarrow C, \quad L \overset{\sim}{\hookleftarrow} K^3, \quad 0 \leftarrow K,$$

where the first and last are simple, the first two are projective, and the last two are injective.

Their endomorphism algebras are, respectively, L, K, L, K, L, K. Their images in the Grothendieck are, respectively,

$$(1,0), \quad (1,1), \quad (2,3), \quad (1,2), \quad (1,3), \quad (0,1).$$

4. Suppose $n = 4$. Can you describe the isomorphism classes of indecomposable representations of type $(L, K^2; \theta)$? What is the connection to the Möbius transformation? Can you prove that there are infinitely many classes, when K is an infinite field?