Representations of hereditary algebras
Exercises 4

1. Let K/k be a field extension of degree n, and consider the finite dimensional hereditary k-algebra

$$R = \begin{pmatrix} K & & \\ K & k & \\ K & k & k \end{pmatrix}.$$

We can write this as $R = A \oplus J$ where A is semisimple and J is a nilpotent two-sided ideal (the Jacobson radical). What is A? What is J? What is J^2?

Use this information to compute the matrix representing the Euler form, the classes $p_i = [P_i]$ of the indecomposable projectives, the classes $q_i := [I_i]$ of the indecomposable injectives, and the matrices representing the Coxeter transformation and its inverse.

We draw the classes of the indecomposable preprojectives and postinjectives as

$$\begin{tikzcd}
& & p_3 \arrow[dr] & & c^{-1}(p_3) & & & & & c(q_3) \arrow[dr] & & q_3 \\
& p_2 \arrow[ur] \arrow[dr] & & c^{-1}(p_2) \arrow[ur] & & \cdots & \cdots & & c(q_2) \arrow[ur] \arrow[dr] & & q_2 \arrow[ur] & \\
p_1 \arrow[ur] & & c^{-1}(p_1) \arrow[ur] & & & & & c(q_1) \arrow[ur] & & q_1 \arrow[ur] & &
\end{tikzcd}$$

Compute these classes when $n = 1, 2$.

Sketch what happens for $n = 3$. Can you prove that there are infinitely many classes of indecomposable preprojectives, and similarly postinjectives? Show that the Coxeter transformation has 1 as an eigenvalue. Compute a corresponding eigenvector. What is the connection to the classes of preprojectives/postinjectives?

Can you find a formula for $\frac{1}{d_i}\big(c^{-r}(p_i) + c^{-r-1}(p_i)\big)$ involving the classes of the other preprojectives, where $d_i := \langle p_i, p_i \rangle$.

2. Give an example of a finite dimensional hereditary tensor algebra $R = T_A(M)$ such that the Grothendieck group is $\mathbb{Z}^4$ with Euler form given by the matrix

$$\langle -, - \rangle \leftrightarrow \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -2 & 2 & \\ & & -2 & 2 \end{pmatrix}.$$

Compute the classes p_i and q_i, the matrix representing the Coxeter transformation. Thus write down all the classes of indecomposable preprojectives, drawing them as

$$\begin{tikzcd}
 & & & p_4 \arrow[dr] & & c^{-1}(p_4) & \\
 & & p_3 \arrow[ur] \arrow[dr] & & c^{-1}(p_3) \arrow[ur] & & \cdots \\
 & p_2 \arrow[ur] \arrow[dr] & & c^{-1}(p_2) \arrow[ur] & & & \\
p_1 \arrow[ur] & & c^{-1}(p_1) \arrow[ur] & & & &
\end{tikzcd}$$

Again, can you find a formula for $\frac{1}{d_i}\big(c^{-r}(p_i) + c^{-r-1}(p_i)\big)$, where $d_i := \langle p_i, p_i \rangle$.

3. Consider the quiver

$$Q\colon \qquad \begin{tikzcd}
 & & & 1 \\
4 & 5 \arrow[l] \arrow[r] & 3 \arrow[ur] \arrow[dr] & \\
 & & & 2
\end{tikzcd}$$

Express the path algebra kQ as a matrix algebra (a subalgebra of $\mathbb{M}_5(k)$). Compute the matrix representing the Euler form.

Draw the classes of the indecomposable preprojectives as

$$\begin{tikzcd}
p_1 \arrow[dr] & & c^{-1}(p_1) \arrow[dr] & & \\
p_2 \arrow[r] & p_3 \arrow[ur] \arrow[r] \arrow[dr] & c^{-1}(p_2) \arrow[r] & c^{-1}(p_3) \arrow[dr] & \cdots \\
 & & p_5 \arrow[ur] \arrow[dr] & & c^{-1}(p_5) \\
 & p_4 \arrow[ur] & & c^{-1}(p_4) \arrow[ur] &
\end{tikzcd}$$