Non-commutative Algebra, SS 2019

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Exercises 1

1. Let K be a field and R a K-algebra.

   (a) Assume that $d := \dim_K R$ is finite. Show that there is an injective K-algebra homomorphism $\mathrm{End}_R(R) \rightarrowtail \mathrm{End}_K(R) \cong \mathbb{M}_d(K)$.

   Since $R \cong \mathrm{End}_R(R)^{\mathrm{op}}$ and $\mathbb{M}_d(K)^{\mathrm{op}} \cong \mathbb{M}_d(K)$, this shows that every finite dimensional K-algebra can be realised as a subalgebra of a matrix algebra. It is analogous to the result that every finite group G can be realised as a subgroup of the symmetric group S_n for $n = |G|$.

   Apply this result to the quaternions $\mathbb{H}$ as an $\mathbb{R}$-algebra, and hence realise them as a subalgebra of $\mathbb{M}_4(\mathbb{R})$.

   (b) More generally, if $S \leq R$ is a subalgebra, then there is an injective K-algebra homomorphism $\mathrm{End}_R(R) \rightarrowtail \mathrm{End}_S(R)$. Apply this to the subalgebra $\mathbb{C} \leq \mathbb{H}$ with $\mathbb{R}$-basis $\{1, i\}$, and hence realise $\mathbb{H}$ as an $\mathbb{R}$-subalgebra of $\mathbb{M}_2(\mathbb{C})$.

   Hint: as left $\mathbb{C}$-modules we have $\mathbb{H} \cong \mathbb{C}^2$ with basis $\{1, j\}$.

2. Let G be a finite group, and K a field of characteristic not dividing $|G|$.

   (a) Show that the following element of KG is an idempotent

   $$e := \frac{1}{|G|} \sum_{g \in G} g.$$

   (b) Let V, W be KG-modules, and $f\colon V \to W$ a K-linear map. Show that the following map is a KG-module homomorphism

   $$\phi(f) : v \mapsto \frac{1}{|G|} \sum_{g \in G} g f(g^{-1} v).$$

   (c) Let $U \leq V$ be KG-modules. As K-vector spaces we can write $V = U \oplus U'$, and hence there exists an idempotent K-linear endomorphism f of V with image U. Show that the KG-linear map $\phi(f)$ is again idempotent with image U. Thus we can write $V = U \oplus U''$ as KG-modules, and hence every KG-module is semisimple.

3. Let K be a field, and V a K-vector space. We know that $K[x]$-module structures on V are in bijection with K-algebra homomorphisms $K[x] \to \mathrm{End}_K(V)$, which in turn are in bijection with (set-theoretic) maps $\{x\} \to \mathrm{End}_K(V)$. In other words, we can regard a $K[x]$-module as a pair (V, T) consisting of a K-vector space V together with a K-linear endomorphism T of V, in which case $x^n \cdot v := T^n(v)$ for all $v \in V$.

(a) Describe the space of $K[x]$-module homomorphisms from (V,T) to (W,U) as a subspace of $\mathrm{Hom}_K(V,W)$.

(b) Show that there exists a short exact sequence of $K[x]$-modules

$$0 \to (K,0) \to (K^2,T) \to (K,0) \to 0, \quad \text{where } T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Is this sequence split?

(c) Given a $K[x]$-module (V,T) and an element $\lambda \in K$, show that the eigenspace $\mathrm{Ker}(T-\lambda) \leq V$ is a semisimple $K[x]$-submodule of (V,T).

(d) Show that for n sufficiently large there exists a decomposition $(V,T) = \mathrm{Ker}(T-\lambda)^n \oplus C$ as $K[x]$-modules.

Hint: write the minimal polynomial of T as $m(t) = (t-\lambda)^n f(t)$ such that $f(\lambda) \neq 0$, so $(t-\lambda)$ and $f(t)$ are coprime. It follows that there exists some polynomials $a(t), b(t) \in K[t]$ such that $a(t)(t-\lambda)^n + b(t)f(t) = 1$. Set $\theta := b(T)f(T) \in \mathrm{End}_K(V)$. Show that θ is a $K[x]$-module homomorphism, that it is idempotent, and that it has image $\mathrm{Ker}(T-\lambda)^n$.

4. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module.

(a) Prove that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module.

(b) Prove that $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module.

(c) Show that every short exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Q} \xrightarrow{f} M \to N \to 0$ splits.

Hint: consider pairs (L,r) consisting of a submodule $L \leq M$ containing $\mathbb{Q}$ and a module homomorphism $r\colon L \to \mathbb{Q}$ with $rf = 1_{\mathbb{Q}}$. This is a poset via $(L,r) \leq (L',r')$ provided $L \leq L'$ and $r = r'|_L$, the restriction of R' to L. Thus by Zorn's Lemma there exists a maximal element (L,r). Now prove that $L = M$.

To be handed in by 18th April.