Non-commutative Algebra, SS 2019

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Exercises 10

1. Some counter-examples.
   (a) Give an example of a *non-split* short exact sequence $0 \to X \to Y \to Z \to 0$ in an abelian category such that $Y \cong X \oplus Z$.
   (b) Give an example of a ring R such that $J(R)$ is a non-zero idempotent ideal.

2. Consider the ring $R := \mathbb{Z}[\sqrt{-5}]$ consisting of all complex numbers of the form $x + y\sqrt{-5}$ with $x, y \in \mathbb{Z}$. Let $I = (2, 1 + \sqrt{-5})$ be the ideal of R generated by 2 and $1 + \sqrt{-5}$.
   (a) Show that I is not principal.
   (b) Use (a), together with the fact that R is a domain, to conclude that I is neither free, nor a summand of R.
   (c) Consider the surjective homomorphism
   $$\pi\colon R^2 \to I, \quad (1,0) \mapsto 2, \quad (0,1) \mapsto 1 + \sqrt{-5}.$$
   Construct a section σ for this map, and hence deduce that I is projective.

> A ring is called hereditary if every ideal is projective. Examples include path algebras of quivers over fields. A commutative domain is hereditary if and only if it is a Dedekind domain. So, in a Dedekind domain, any non-principal ideal is projective but not free.

3. Consider the quiver $\bar{Q}$,
   $$\bar{Q}\colon \quad 1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{b^*}{\overset{b}{\rightleftarrows}} 3$$
   Given $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$, the deformed preprojective algebra Π^λ is the quotient of $\mathbb{C}\bar{Q}$ by the relations
   $$-a^*a = \lambda_1, \quad aa^* - b^*b = \lambda_2, \quad bb^* = \lambda_3.$$
   Assume $\lambda_2 \neq 0$, and consider a Π^λ-representation
   $$X\colon \quad X_1 \underset{X_{a^*}}{\overset{X_a}{\rightleftarrows}} X_2 \underset{X_{b^*}}{\overset{X_b}{\rightleftarrows}} X_3$$
   (a) Set
   $$\alpha := \begin{pmatrix} X_{a^*} \\ -X_b \end{pmatrix} \colon X_2 \to X_1 \oplus X_3 \quad \text{and} \quad \beta := \tfrac{1}{\lambda_2}\big(X_a, X_{b^*}\big)\colon X_1 \oplus X_3 \to X_2.$$
   Compute $\beta\alpha$, and hence deduce that $\alpha\beta$ is an idempotent endomorphism of $X_1 \oplus X_3$.

(b) Now consider the $\bar{Q}$-representation

$$Y\colon \quad Y_1 \underset{Y_{a^*}}{\overset{Y_a}{\rightleftarrows}} Y_2 \underset{Y_{b^*}}{\overset{Y_b}{\rightleftarrows}} Y_3$$

where $Y_1 = X_1$, $Y_2 := \mathrm{Im}(1-\alpha\beta)$ and $Y_3 = X_3$, with maps

$$Y_a := \lambda_2(1-\alpha\beta)\iota_1, \quad Y_{a^*} := -\pi_1, \quad Y_b := \pi_3, \quad Y_{b^*} := \lambda_2(1-\alpha\beta)\iota_3.$$

Here, ι_i and π_i are the usual inclusions and projections associated to the direct sum $X_1 \oplus X_3$.

Show that Y is a Π^μ-representation, for $\mu = (\lambda_1+\lambda_2, -\lambda_2, \lambda_2+\lambda_3)$.

(c) Show that $X \mapsto Y$ induces a functor $F\colon \Pi^\lambda\text{–Mod} \to \Pi^\mu\text{–Mod}$.

(d) The analogous construction starting from a Π^μ-representation yields a functor $G\colon \Pi^\mu\text{–Mod} \to \Pi^\lambda\text{–Mod}$. Prove that $GF \cong \mathrm{id}$. Analogously, $FG \cong \mathrm{id}$, so that $\Pi^\lambda\text{–Mod} \cong \Pi^\mu\text{–Mod}$. In other words, Π^λ and Π^μ are Morita equivalent.

4. Let $\mathcal{J}$ and $\mathcal{C}$ be categories, with $\mathcal{J}$ small.

(a) Show that the hom functor preserves limits. In other words, given a functor $F\colon \mathcal{J} \to \mathcal{C}$ and an object $X \in \mathcal{C}$, we have a functor $\mathrm{Hom}_\mathcal{C}(X, F)\colon \mathcal{J} \to \mathbf{Set}$. Show that, if $\lim F \in \mathcal{C}$ exists, then we have a natural isomoprhism $\mathrm{Hom}_\mathcal{C}(X, \lim F) \cong \lim \mathrm{Hom}_\mathcal{C}(X, F)$.

Since the colimit of $G\colon \mathcal{J} \to \mathcal{C}$ is the limit of $G^{\mathrm{op}}\colon \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$, the dual statement is that $\mathrm{Hom}_\mathcal{C}(\operatorname{colim} G, Y) \cong \lim \mathrm{Hom}_\mathcal{C}(G, Y)$.

(b) Prove that colimits commute with tensor products (in module categories). In other words, given a functor $G\colon \mathcal{J} \to R\text{–Mod}$ and a bimodule ${}_SM_R$, we have a functor $M \otimes_R G\colon \mathcal{J} \to S\text{–Mod}$. If $\operatorname{colim} G \in R\text{–Mod}$ exists, then $\operatorname{colim}(M \otimes_R G) \in S\text{–Mod}$ exists, and we have a natural isomorphism $M \otimes_R \operatorname{colim} G \cong \operatorname{colim}(M \otimes_R G)$.

Hint: use the tensor-hom adjointness, together with (a).

(c) Give an example to show that limits do not commute with tensor products.

Hint: consider $R = S = \mathbb{Z}$, $M = \mathbb{Q}$, and the functor $\mathbb{N}^{\mathrm{op}} \to \mathbf{Ab}$, $n \mapsto \mathbb{Z}/(p^n)$ for some fixed prime p.

To be handed in by 28th June.