Non-commutative Algebra, SS 2019

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Exercises 11

1. Some injective envelopes.

   (a) Let ${}_{\mathbb{Z}}M$ be a torsion-free abelian group, so $0 \neq x \in M$ and $nx = 0$ implies $n = 0$. Show that $M \to \mathbb{Q} \otimes_{\mathbb{Z}} M$ is an injective envelope.

   (b) More generally, let R be a left Noetherian domain (so $0 \neq b \in R$ and $ab = 0$ implies $a = 0$), and let ${}_RM$ be a torsion-free R-module (so $0 \neq x \in M$ and $ax = 0$ implies $a = 0$). Show that $M \to R_S \otimes_R M$ is an injective envelope, where $S = R - \{0\}$.

   Hint: Let $I \leq R$ be a left ideal, and consider $f\colon I \to R$. If $0 \neq x \in I$, we can write $f(x) = xm_x$ for some $m_x \in S^{-1}M$. Show that $m_x = m_y$ for all $x, y \in I$.

2. Injective envelopes as weak initial or terminal objects.

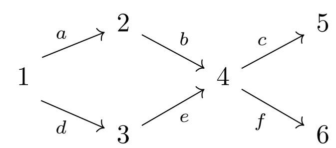
   Let R be a ring and M an R-module. We will consider pairs (X, x) such that X is an R-module and $x\colon M \rightarrowtail X$ is an injective R-module homomorphism. We say that (X, x) is essential provided x is an essential extension, and say that (X, x) is injective provided X is an injective R-module. Prove that the following are equivalent.

   (a) (I, i) is an injective envelope of M.

   (b) (I, i) is essential, and for any other essential (X, x), there exists an injective map $f\colon X \to I$ with $fx = i$.

   (c) (I, i) is injective, and for any other injective (X, x), there exists an injective map $f\colon I \to X$ such that $fi = x$.

3. Consider the algebra $A := KQ/I$ for the quiver

   Q:
   

   and admissible ideal $I = (ba - ed, cb, fe)$.

   (a) Compute the indecomposable projectives $P[i]$ and their radicals.

   (b) Compute the indecomposable injectives $I[i]$ and their quotients $I[i]/S[i]$.

   (c) Compute the minimal projective resolutions of the simple modules.

   (d) Compute the minimal injective resolutions of the projective modules.

A minimal projective resolution of a module X is an exact sequence

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0,$$

such that each $d_i \colon P_i \to \mathrm{Ker}(d_{i-1})$ is a projective cover. Dually, a minimal injective coresolution of X is an exact sequence

$$0 \to X \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \xrightarrow{d^2} I^2 \to \cdots$$

such that each $d^i \colon \mathrm{Im}(d^{i-1}) \to I^i$ is an injective envelope.

4. Colimits of flat modules.

   Fix a K-algebra R. Recall that, given a directed system of right R-modules $f_i \colon M_i \to M_{i+1}$ for $i \geq 1$, its colimit is

$$\operatorname{colim} M_i := \bigoplus_i M_i / U$$

   where U is the submodule of $\bigoplus_i M_i$ generated by all $m_i - f_i(m_i)$.

   For convenience we define $f_{ji} := f_{j-1} \cdots f_i \colon M_i \to M_j$ for all $j \geq i$, so that $f_{ii} = \mathrm{id}_{M_i}$ and $f_{i+1,i} = f_i$.

   (a) Suppose $R = \mathbb{Z}$, $M_i := \frac{1}{i!}\mathbb{Z} \leq \mathbb{Q}$, and $f_i \colon M_i \rightarrowtail M_{i+1}$ is the usual inclusion map. Show that $\operatorname{colim} M_i = \mathbb{Q}$.

   (b) Let (M_i, f_i) be a directed system of right R-modules. Show that every element in $\operatorname{colim} M_i$ is the image of some $m_j \in M_j$. Show further that if $m_j \in M_j$ is sent to zero in $\operatorname{colim} M_i$, then $f_{kj}(m_j) = 0$ for some $k \geq j$.

   (c) Given $g \colon X \to Y$, suppose that $\mathrm{id} \otimes g \colon M_i \otimes_R X \to M_i \otimes_R Y$ is injective for all i. Show that the map $\mathrm{id} \otimes g \colon (\operatorname{colim} M_i) \otimes_R X \to (\operatorname{colim} M_i) \otimes_R Y$ is also injective.

   (d) Deduce that if M_i is flat for all i, then $\operatorname{colim} M_i$ is again flat. In particular, a directed colimit of projective modules is always flat, but as part (a) showed, it is not necessarily projective.

   > There is a more general construction of directed colimits using directed posets, not just the natural numbers. In this more general setting, Lazard's Theorem tells us that every flat module is a directed colimit of finitely generated free modules.

To be handed in by 5th July.