Non-commutative Algebra, SS 2019

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Exercises 2

Throughout, K is a field.

1. (a) Set $R := K[x_2, x_3, x_4, \ldots]/(x_2^2, x_3^3, x_4^4, \ldots)$ and $I := (x_2, x_3, x_4, \ldots)$, an ideal in R. Show that I is a nil ideal, but is not a nilpotent ideal.

   (b) Compute the (two-sided) ideal $I \lhd \mathbb{M}_2(K)$ generated by the nilpotent element $e^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2. Consider the vector space $V := K^{(\mathbb{N})}$, with basis $e_1, e_2, e_3, \ldots$. Let R be the set of those K-linear endomorphisms θ of V such that $\theta(e_i) \in \operatorname{span}\{e_1, \ldots, e_i\}$ and $\theta(e_n) \in \operatorname{span}\{e_n\}$ for almost all n.

   (a) Show that R is a K-algebra, which we may regard as upper triangular matrices in $\mathbb{M}_\infty(K)$ with only finitely many off-diagonal entries.

   (b) Show that $\theta \in R$ is invertible if and only if all the diagonal entries of θ are invertible.

   (c) Show that $J(R)$ consists of the strictly upper triangular matrices, so those having zero on the diagonal.

   (d) Show that $J(S)$ is a nil ideal, but is not a nilpotent ideal.

3. Let R be a ring, and M an R-module. We set $\operatorname{rad}(M)$ to be the intersection of all maximal submodules. (Note: M may not have any maximal submodules, in which case $\operatorname{rad}(M) = M$.)

   (a) Show that $M/\operatorname{rad}(M)$ embeds into a product of simple modules, and hence that $J(R)M \subset \operatorname{rad}(M)$.

   (b) Show that if $R/J(R)$ is semisimple, then $\operatorname{rad}(M/J(R)M) = 0$, and hence that $J(R)M = \operatorname{rad}(M)$.

   (c) Give an example of a ring R and a finitely generated R-module M for which $J(R)M \neq \operatorname{rad}(M)$.

4. Let R be a finite dimensional K-algebra. A bilinear form $\langle -, - \rangle \colon R \times R \to K$ is associative provided $\langle xy, z \rangle = \langle x, yz \rangle$ for all $x, y, z \in R$.

   (a) Show that there is a bijection between associative bilinear forms and elements of $\mathrm{Hom}_K(R, K)$, sending $\langle -, - \rangle$ to the map $x \mapsto \langle x, 1 \rangle$.

   (b) Show that R admits a non-degenerate associative bilinear form if and only if there is an isomorphism of (left) R-modules $R \xrightarrow{\sim} \mathrm{Hom}_K(R, K)$. In this case we call R a Frobenius algebra.

   Note that the R-module structure on $\mathrm{Hom}_K(R, K)$ is given by $(rf)(s) := f(sr)$ for all $r, s \in R$ and $f \in \mathrm{Hom}_K(R, K)$. Also, $\langle -, - \rangle$ is non-degenerate provided $\langle r, - \rangle = 0$ implies $r = 0$ and $\langle -, s \rangle = 0$ implies $s = 0$.

   (c) We call R a symmetric algebra provided there exists a non-degenerate associative symmetric bilinear form. Show that $K[x]/(x^n)$ is a symmetric algebra. Show that $K[x, y]/(x^2, y^2)$ is a symmetric algebra. Show that $K[x, y]/(x, y)^2$ is not a Frobenius algebra.

   (d) Let R be a division algebra. Show that, in characteristic zero, the bilinear form arising from the character $R \rightarrowtail \mathrm{End}_K(R) \xrightarrow{\mathrm{tr}} K$ is non-degenerate, and hence that R is symmetric. In fact, every semisimple algebra is a symmetric algebra.

To be handed in early on 29th April.