Non-commutative Algebra, SS 2019

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Exercises 3

1. Let K be a field, $R = \mathbb{M}_n(K)$, and $M = K^n$ with its natural left R-module.

   (a) Let θ be a K-algebra automorphism of R. Show that the assignment $R \times M \to M$, $(r, m) \mapsto \theta(r)m$, determines a new R-module structure on M. We denote the resulting R-module by ${}_\theta M$.

   (b) Explain why $M \cong {}_\theta M$ as R-modules. (Hint: R is a semisimple algebra.)

   (c) Deduce that there exists $\phi \in \mathrm{Aut}_K(M)$ such that $\theta(r) = \phi r \phi^{-1}$.

   Thus $\phi \in R$ is a unit, and θ is an inner automorphism. This is a special case of the Noether-Skolem Theorem.

2. This exercise shows that tensor products of semisimple algebras need not be semisimple.

   (a) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebras. Find explicit descriptions of the idempotents in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ corresponding to $(1, 0)$ and $(0, 1)$ in $\mathbb{C} \times \mathbb{C}$.

   This exercise can be generalised: Let K/k be a finite Galois extension with Galois group G. Then $K \otimes_k K \cong K^{|G|}$.

   Hint: Use the Chinese Remainder Theorem. If you want to do the generalisation, you will also need the Primitive Element Theorem.

   (b) Let K be a field of characteristic $p > 0$, and suppose $x \in K$ is not a p-th power in K (so is not of the form y^p for some $y \in K$). Set $L := K[t]/(t^p - x)$, a field extension of degree p.

   Show that $L \otimes_K L$ contains a non-zero element u satisfying $u^p = 0$. Moreover, $L \otimes_K L \cong L[u]/(u^p)$. In particular, $L \otimes_K L$ is not semisimple.

3. Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of quaternions.

   (a) For $q \in \mathbb{H}$ denote by $\lambda(q), \rho(q) \in \mathrm{End}_{\mathbb{R}}(\mathbb{H})$ left and right multiplication by q. Show that there is an isomorphism of $\mathbb{R}$-algebras

   $$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{\mathbb{R}}(\mathbb{H}), \quad q \otimes q' \mapsto \lambda(q)\rho(q').$$

   (b) Recall that $\mathbb{H}$ is a left $\mathbb{C}$-module with basis $\{1, j\}$, and we have an injective $\mathbb{R}$-algebra homomorphism

   $$\mathbb{H} \rightarrowtail \mathbb{M}_2(\mathbb{C}), \quad z + wj \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad z, w \in \mathbb{C}.$$

   Show that this induces an isomorphism of $\mathbb{C}$-algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{M}_2(\mathbb{C})$.

Hint: By dimension arguments it is enough to prove one of injectivity or surjectivity.

4. Let R be a K-algebra, X a right R-module and Y a left R-module. Recall that $X \otimes_R Y$ is defined to be the quotient of the free abelian group F on symbols $x \otimes y$ for $x \in X$ and $y \in Y$, subject to the relations

$$(x + x') \otimes y = x \otimes y + x' \otimes y, \quad x \otimes (y + y') = x \otimes y + x \otimes y'$$

$$(xr) \otimes y = x \otimes (ry) \text{ for all } r \in R.$$

(a) Show that $X \otimes_R Y$ is a K-module. (In particular, you need to check that the action is well-defined.)

(b) Observe that the map $\iota\colon X \times Y \to X \otimes_R Y$, $(x, y) \mapsto x \otimes y$, is K-bilinear and R-balanced. Now let Z be any K-module. Prove that the assignment $\theta \mapsto \theta\iota$ induces a bijection between $\mathrm{Hom}_K(X \otimes_R Y, Z)$ and K-bilinear R-balanced maps $X \times Y \to Z$.

Hint: For the surjectivity, suppose we are given a K-bilinear R-balanced map $\phi\colon X \times Y \to Z$. Construct a linear map $F \to Z$, and deduce that this determines a K-linear map $X \otimes_R Y \to Z$.

(c) We have shown that the pair $(X \otimes_R Y, \iota)$ satisfies the following universal property:

- $X \otimes_R Y$ is a K-module, and the map $\iota\colon X \times Y \to X \otimes_R Y$ is K-bilinear and R-balanced.
- If Z is a K-module and $\phi\colon X \times Y \to Z$ is K-bilinear and R-balanced, then there exists a unique K-linear map $\theta\colon X \otimes_R Y \to Z$ satisfying $\phi = \theta\iota$.

Suppose (M, j) also satisfies this universal property. Show that there is a unique K-linear map $\theta\colon X \otimes_R Y \to M$, and that this is necessarily an isomorphism.

To be handed in by 3rd May.