Non-commutative Algebra, SS 2019

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 5

Throughout, K is a field.

1. Let A be the quotient of $K\langle x, y\rangle$ subject to the relations

$$x^2 = x + 1, \quad xy = y, \quad yx = y.$$

Determine which of the following overlap ambiguities satisfy the diamond condition:

$$x^2 \cdot x = x \cdot x^2, \quad yx \cdot x = y \cdot x^2, \quad xy \cdot x = x \cdot yx.$$

Are there any other ambiguities?

2. Let A be the quotient of $K\langle x, y\rangle$ subject to the relations

$$x^2 = 0, \quad y^2 = 0, \quad yxy = -xyx.$$

There are five ambiguities (all of them overlap ambiguities):

$$x^2 \cdot x = x \cdot x^2, \quad y^2 \cdot y = y \cdot y^2$$

$$y^2 \cdot xy = y \cdot yxy, \quad yx \cdot y^2 = yxy \cdot y, \quad yxy \cdot xy = yx \cdot yxy.$$

Verify the diamond condition. Hence write out a basis for A and compute $\dim A$.

3. Consider the algebra $R_n = KQ_n/(S_n)$, where Q_n is the quiver

$$1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_2}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_3}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-1}}{\rightleftarrows}} n$$

and the relations S_n are

$$b_1 a_1 = 0, \quad b_i a_i = a_{i-1} b_{i-1} \text{ for } 1 < i < n, \quad a_{n-1} b_{n-1} = 0.$$

We use the standard ordering on the vertices, and order the arrows as

$$a_1 < \cdots < a_n < b_1 < \cdots < b_n$$

(a) Show that there are two ambiguities, and that the Diamond Lemma fails for both of them.

(b) Set $a(i; r) := a_{i+r-1} \cdots a_i$, a path of length r starting at i, and similarly $b(i; s) := b_i \cdots b_{i+s-1}$, a path of length s ending at i. In particular $a(i; 0) = e_i = b(i; 0)$. Show that R_n is spanned by the (images of the) set

$$\mathcal{B}' := \{p(i; r, s) : 1 \le i \le n,\ 0 \le r, s \le n - i\},$$

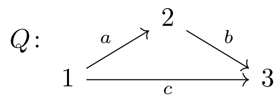
where $p(i; r, s) := a(i; r) b(i; s)$.

(c) Use the relations to show that $p(i;r,s) = 0$ whenever $i + r + s = n + 1$.

(d) Adjoining the relations from (c) to S_n, find all ambiguities and hence show that the Diamond Lemma holds.

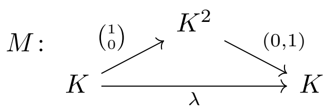
(e) Find a basis $\mathcal{B} \subset \mathcal{B}'$ of R_n, and hence compute $\dim R_n$.

4. Consider the quiver

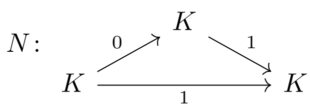


(a) For each $i = 1, 2, 3$ compute the K-representation P_i corresponding to the KQ-module KQe_i.

(b) For $\lambda \in K$ consider the K-representations



and



Construct surjective homomorphisms

$$\theta\colon P_1 \oplus P_2 \to M \quad \text{and} \quad \phi\colon P_1 \oplus P_2 \to N.$$

Give an isomorphism from $\mathrm{Ker}(\theta)$ to a direct sum of the P_i, and similarly for $\mathrm{Ker}(\phi)$.

To be handed in by 17th May.