Non-commutative Algebra, SS 2019

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Exercises 6

1. Let K be a field with $\operatorname{char} K \neq 2$. For $a_1, \ldots, a_n \in K$, let $C_K(a_1, \ldots, a_n)$ be the Clifford algebra with respect to the quadratic form $q(\sum_i \lambda_i x_i) := \sum_i a_i \lambda_i^2$. Thus
$$C_K(a_1, \ldots, a_n) := K\langle x_1, \ldots, x_n \rangle / (x_i^2 - a_i, x_i x_j + x_j x_i).$$
This has dimension 2^n, with basis given by (the images of) the elements $x_{i_1} \cdots x_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$.

   (a) Show that $C_K(1) \cong K \times K$ and over the real numbers we have $C_{\mathbb{R}}(-1) \cong \mathbb{C}$.

   (b) Show that $C_K(1, a) \cong \mathbb{M}_2(K)$ for all $a \neq 0$.
   Hint: consider the matrices
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}.$$

   (c) Show that over the real numbers $C_{\mathbb{R}}(1, 1, 1) \cong \mathbb{M}_2(\mathbb{C})$.
   Hint: consider the Pauli spin matrices
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Hilbert's Basis Theorem fails if σ is not invertible.

   Let $R = K[t]$ be a polynomial ring over a field K, and let σ be the K-algebra endomorphism of R sending $t \mapsto t^2$. We claim that the skew-polynomial ring $R[x; \sigma]$ is neither left nor right Noetherian.

   (a) Consider the left ideal I_n generated by $t, tx, \ldots, tx^n$. Show that tx^{n+1} is not contained in I_n. Deduce that the left ideal generated by all tx^i for $i \geq 0$ is not finitely generated.

   (b) Consider the right ideal J_n generated by $t^n x^n$. Show that this has K-basis given by the $t^i x^j$ such that $i \equiv n \bmod 2^n$ and $j \geq n$. Deduce that the right ideal generated by all $t^n x^n$ with $n \geq 2$ is not finitely generated.

3. Skew polynomial rings for inner derivations.

   Consider a skew-polynomial ring $R[x; \sigma, \delta]$. Assume that $\delta \colon R \to {}_\sigma R$ is an inner derivation, so there exists $m \in R$ such that $\delta(r) = \sigma(r)m - mr$ for all $r \in R$. Show that $R[x; \sigma, \delta]$ is isomorphic to the skew-polynomial ring $R[y; \sigma]$, via the map $r \mapsto r$ for $r \in R$ and $x \mapsto y - m$.

   As a special case, if $\sigma = \mathrm{id}_R$ and δ is an inner derivation, then the skew-polynomial ring $R[x; \delta]$ is isomorphic to an ordinary polynomial ring $R[y]$.

4. Let R be a ring and S a multiplicative subset satisfying the conditions

left Ore: given $(s,r) \in S \times R$ there exists $(s',r') \in S \times R$ with $s'r = r's$.

left reversible: given $r \in R$, if $rs = 0$ for some $s \in S$, then $s'r = 0$ for some $s' \in S$.

Let M be a left R-module, and define a relation $\sim$ on $S \times M$ via

$(s,m) \sim (s',m')$provided there exist $r, r' \in R$ with $rm = r'm'$ and $rs = r's' \in S$.

This is clearly reflexive and symmetric. Here we prove that it is transitive, and hence an equivalence relation. Suppose therefore that

$$(s,m) \sim (s',m') \sim (s'',m'')$$

so that we have $a, a', b', b'' \in R$ such that

$$am = a'm', \quad b'm' = b''m'', \quad as = a's' \in S, \quad b's' = b''s'' \in S.$$

We need to find $c, c'' \in R$ such that

$$cm = c''m'', \quad cs = c''s'' \in S. \qquad (\dagger)$$

(a) Assume we have found $t, t'' \in R$ such that $ta' = t''b'$ and one of t, t'' lies in S. Use these elements to construct c, c'' satisfying $(\dagger)$.

(b) Apply the left Ore condition to the pair $(a's', b's') \in S \times R$.

(c) Now apply the left reversibility condition to your answer to obtain t, t'' as in (1).

To be handed in by 24th May.