Non-commutative Algebra, SS 2019

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 7

1. Consider the algebra $U_v(\mathfrak{sl}_2)$ over $\mathbb{C}$, having generators k, k^{-1}, e, f subject to the relations

$$kk^{-1} = 1 = k^{-1}k, \quad kek^{-1} = v^2 e, \quad kfk^{-1} = v^{-2} f, \quad ef - fe = \frac{k - k^{-1}}{v - v^{-1}},$$

where $0 \neq v \in \mathbb{C}$ is not a root of unity.

   (a) Show that $U_v(\mathfrak{sl}_2)$ is spanned by the elements $e^a k^r f^b$ for $a, b \geq 0$ and $r \in \mathbb{Z}$.

   (b) Given $\lambda \in \mathbb{C}$, show that $\mathbb{C}[t]$ becomes a $U_v(\mathfrak{sl}_2)$-module via the action

$$k \cdot t^n := \lambda v^{2n} t^n, \qquad e \cdot t^n := t^{n+1},$$
$$k^{-1} \cdot t^n := \lambda^{-1} v^{-2n} t^n, \qquad f \cdot t^n := -[n][\lambda; n-1] t^{n-1},$$

   where

$$[\lambda; n] := \frac{\lambda v^n - \lambda^{-1} v^{-n}}{v - v^{-1}} \quad \text{and} \quad [n] := [1; n].$$

   This is called the Verma module, denoted V^λ.

   Hint: the action determines a module structure for the free algebra with generators k, k^{-1}, e, f, so we just need to check that all the relations hold, so for example that $kek^{-1} \cdot t^n = v^2 e \cdot t^n$ for all n.

2. For a field K we set $K[\epsilon] := K[x]/(x^2)$, where ϵ is the image of x. Then, as a K-vector space, we have $\mathbb{M}_n(K[\epsilon]) = \mathbb{M}_n(K) + \mathbb{M}_n(K)\epsilon$.

   (a) We can write $\mathrm{GL}_n(R) = \{M \in \mathbb{M}_n(R) : \det M \in R^\times\}$. Show that

$$\{M \in \mathbb{M}_n(K) : I + M\epsilon \in \mathrm{GL}_n(K[\epsilon])\} = \mathbb{M}_n(K).$$

   (b) We can write $\mathrm{SL}_n(R) = \{M \in \mathbb{M}_n(R) : \det(M) = 1\}$. Show that

$$\{M \in \mathbb{M}_n(K) : I + M\epsilon \in \mathrm{SL}_n(K[\epsilon])\} = \{M \in \mathbb{M}_n(K) : \mathrm{tr}(M) = 0\}.$$

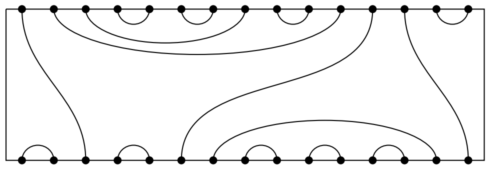
   (c) Set $\mathrm{SO}_n(R) = \{M \in \mathbb{M}_N(R) : \det(M) = 1,\ M^t M = I\}$. Show that, in characteristic different from 2,

$$\{M \in \mathbb{M}_n(K) : I + M\epsilon \in \mathrm{SO}_n(K[\epsilon])\} = \{M \in \mathbb{M}_n(K) : M + M^t = 0\}.$$

In other words, we have the following descriptions of the Lie algebras

$$\mathfrak{gl}_n = \mathbb{M}_n(K), \quad \mathfrak{sl}_n = \{M : \mathrm{tr}(M) = 0\}, \quad \mathfrak{so}_n = \{M : M + M^t = 0\}.$$

3. Recall that the Temperley-Lieb algebra $TL_n(\delta)$ has a basis indexed by diagrams of the form



so having n dots on the top row, n dots on the bottom row, and the dots are connected by non-crossing lines. (In the lectures the diagrams were drawn vertically, rather than horizontally.)

We wish to compute $C_n := \dim TL_n(\delta)$.

(a) Show that the C_n satisfy the recursion $C_0 = 1$ and $C_{n+1} = \sum_{a+b=n} C_a C_b$. Hint: We label the vertices clockwise from 1 to $2n$. Given a diagram, say with vertex 1 is connected to vertex i. Show that $i = 2a + 2$ for some $0 \le a < n$, and that we have two smaller diagrams on vertices $2, \dots, 2a+1$ and on $2a + 3, \dots, 2n$.

(b) Consider the generating function $C(t) := \sum_{n\ge 0} C_n t^n$. Show that the recursion corresponds to the functional equation $C(t) = 1 + tC(t)^2$. Show further that this has solution $C(t) = (1 \pm \sqrt{1-4t})/2t$.

(c) Show that the Taylor expansion at $t = 0$ is

$$\sqrt{1-4t} = 1 - 2t \sum_{n\ge 0} \frac{(2n)!}{n!(n+1)!} t^n.$$

Hence $(1 - \sqrt{1-4t})/2t$ does not have a pole at 0, and so its Taylor expansion at $t = 0$

$$C(t) = \sum_{n\ge 0} \frac{(2n)!}{n!(n+1)!} t^n$$

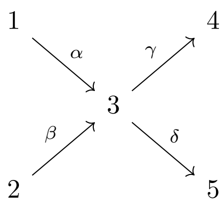
yields a solution to the functional equation $C(t) = 1 + tC(t)^2$.

This proves that

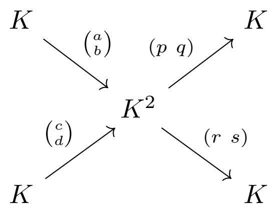
$$\dim TL_n(\delta) = C_n = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n},$$

which is the n-th Catalan number.

4. We consider the following quiver Q of type $\widetilde{\mathbb{D}}_4$



We consider representations of Q over a field K of dimension vector $(1, 1, 2, 1, 1)$. Fixing bases, we can write



which we abbreviate to $\left[\binom{a}{b}, \binom{c}{d}, (p\,q), (r\,s)\right]$.

(a) Show that the representations $\left[\binom{1}{0}, \binom{0}{1}, (p\,q), (r\,s)\right]$ has endomorphism algebra K if and only if at most one of p, q, r, s is zero.

(b) In this case show further that, up to isomorphism, we may assume three of p, q, r, s equal one. Use this to give one representative from each isomorphism class.

(c) There is one other indecomposable (up to isomorphism) of dimension vector $(1, 1, 2, 1, 1)$. Find it.

(d) Exhibit those pairs of non-isomorphic indecomposables M, N for which there exists a non-zero homomorphism $M \to N$.

To be handed in by 31st May.