Non-commutative Algebra, SS 2019

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Exercises 8

1. An initial object in a category $\mathcal{C}$ is an object I such that, for each object X, there exists a unique morphism $\iota_X \colon I \to X$. Dually T is a terminal object provided that, for each X, there exists a unique morphism $\pi_X \colon X \to T$.

   (a) What are the initial and terminal objects of the following categories?

      (i) The category **Set** of sets.

      (ii) The category **Gp** of groups.

      (iii) The category **Ring** of rings.

   (b) Show further that initial objects are unique up to unique isomorphism, so if I' is another initial object, then there exists a unique map $\iota \colon I \to I'$, which is furthermore an isomorphism.

2. Let $\mathcal{C}$ be a category and $F \colon \mathcal{C} \to \mathbf{Set}$ a functor.

   (a) Show that there is a category $\mathrm{el}(F)$ having objects the pairs (X, x) for $X \in \mathcal{C}$ and $x \in F(X)$, and morphisms $f \colon (X, x) \to (Y, y)$ where $f \colon X \to Y$ is a morphism in $\mathcal{C}$ such that $F(f)(x) = y$.

   This is called the category of elements of F, and is an example of a coslice category, or comma category.

   (b) A pair (X, x) satisfies the universal property for F provided that, for all objects $Y \in \mathcal{C}$ and all $y \in F(C)$, there exists a unique morphism $f \colon X \to Y$ sending x to y, so $F(f)(x) = y$. Show that (X, x) satisfies the universal property for F if and only if (X, x) is an initial object in $\mathrm{el}(F)$.

   (c) Yoneda's Lemma tells us that each pair (X, x) determines a natural transformation $\eta_{X,x} \colon \mathrm{Hom}(X, -) \Rightarrow F$. Show that $\eta_{X,x}$ is a natural isomorphism (and so F is a representable functor) if and only if (X, x) is an initial object in $\mathrm{el}(F)$.

   This explains the relationship between representable functors, initial objects, and universal properties.

   (d) An example of this construction is the cokernel. Let $f \colon X \to Y$ be a module homomorphism. The cokernel consists of an object $\mathrm{Coker}(f)$ and a map $g \colon Y \to \mathrm{Coker}(f)$ with $gf = 0$ satisfying the universal property: for all Z and all $h \colon Y \to Z$ with $hf = 0$, there exists a unique morphism $h' \colon \mathrm{Coker}(f) \to Z$ with $h = h'g$.

   What is the corresponding functor we should consider?

   (e) Another example is the pushout of a pair of morphisms $f \colon W \to X$ and $g \colon W \to Y$. Here we consider the functor sending an object Z to the set of morphisms $(g', f') \colon X \oplus Y \to Z$ satisfying $g'f = f'g$. What is the corresponding universal property of the pushout?

3. Let R be a ring.

(a) Show that for R-modules X, Y, Z we have $Z \cong X \oplus Y$ if and only if there exist morphisms

$$X \underset{\pi_X}{\overset{\iota_X}{\rightleftarrows}} Z \underset{\iota_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$$

satisfying

$$\pi_X \iota_X = \mathrm{id}_X, \quad \pi_Y \iota_Y = \mathrm{id}_Y, \quad \iota_X \pi_X + \iota_Y \pi_Y = \mathrm{id}_Z.$$

(b) A functor $F\colon R\text{–Mod} \to \mathbf{Ab}$ from R-modules to abelian groups (that is, $\mathbb{Z}$-modules) is said to be additive provided the map

$$\mathrm{Hom}_R(X, Y) \to \mathrm{Hom}(F(X), F(Y)), \quad f \mapsto F(f),$$

is a homomorphism of additive groups, so $F(f + g) = F(f) + F(g)$.
Show that, if F is additive, then $F(X \oplus Y) \cong F(X) \oplus F(Y)$ for all R-modules X, Y.

(c) Given an R-module X, we set $\mathrm{add}(X)$ to be the full subcategory having those objects Y for which $Y \oplus Z \cong X^n$ for some n and some Z. Let $\eta\colon F \Rightarrow G$ be a natural transformation of additive functors. Show that $\eta_{Y \oplus Z}$ corresponds to the map $\eta_Y \oplus \eta_Z\colon F(Y) \oplus F(Z) \to G(Y) \oplus G(Z)$.

(d) Deduce that if $\eta\colon F \Rightarrow G$ is a natural transformation of additive functors such that η_X is an isomorphism for some X, then η_Y is an isomorphism for all $Y \in \mathrm{add}(X)$.

4. Let R be a ring, and X an R-module.

(a) Show that $F(Y) := \mathrm{Hom}_R(X, R) \otimes_R Y$ and $G(Y) := \mathrm{Hom}_R(X, Y)$ define additive functors $R\text{-Mod} \to \mathbf{Ab}$.

(b) Show that there is a natural transformation $\eta\colon F \Rightarrow G$ such that η_R is an isomorphism.

(c) Deduce that η_P is an isomorphism for all $P \in \mathrm{add}(R)$.
(Note: $\mathrm{add}(R) = R$–proj is the full subcategory of finitely generated projective R-modules.)

To be handed in by 7th June.