Non-commutative Algebra, SS 2019

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Exercises 9

1. We recall the definition of a limit.

   Let $\mathcal{C}$ be a category, and Q a finite quiver. Given an object $X \in \mathcal{C}$, there exists a functor $c_X \colon Q \to \mathcal{C}$ such that $c_X(v) = X$ for all vertices $v \in Q$, and $c_X(a) = \mathrm{id}_X$ for all arrows $a \in Q$. If $\theta \colon X \to Y$ is a morphism in $\mathcal{C}$, then we have a natural transformation $c_\theta \colon c_X \Rightarrow c_Y$ such that $c_\theta(v) = \theta$ for all vertices v. Thus c determines a functor $\mathcal{C} \to \mathrm{Fun}(Q, \mathcal{C})$.

   Given a functor $F \colon Q \to \mathcal{C}$, we have an associated functor $\tilde{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that $\tilde{F}(X) = \mathrm{Nat}(c_X, F)$, the set of natural transformations $c_X \to F$. If $\theta \colon X \Rightarrow Y$ in $\mathcal{C}$, then $\tilde{F}(\theta) \colon \mathrm{Nat}(c_Y, F) \to \mathrm{Nat}(c_X, F)$ is given by composition with c_θ, so sends a natural transformation $\eta \colon c_Y \Rightarrow F$ to ηc_θ.

   We say that F has a limit, denoted $\lim F$, provided $\tilde{F}$ is representable. In other words, there exists a pair (X, ξ) with $X \in \mathcal{C}$ and $\xi \colon c_X \Rightarrow F$ such that, for all other pairs (Y, η), there exists a unique $\theta \colon Y \to X$ with $\eta = \xi c_\theta$.

   We say that $\mathcal{C}$ is finitely complete provided that, for each finite quiver Q, every functor $F \colon Q \to \mathcal{C}$ has a limit.

   Prove that the following are equivalent for $\mathcal{C}$.

   (a) $\mathcal{C}$ is finitely complete.

   (b) $\mathcal{C}$ has all finite products and equalisers.

   (c) $\mathcal{C}$ has all pullbacks and a terminal object.

   Hints. For (b) implies (a), suppose we are given a functor $F \colon Q \to \mathcal{C}$. Set $V := \prod_{v \in Q} F(v)$ and $W := \prod_{a \in Q} F(h(a))$. Show that there are two obvious maps $V \to F(h(a))$, yielding two morphisms $V \to W$. Show that their equaliser is $\lim F$.

   For (c) implies (b), show that $X \times Y$ is the pullback of the unique maps $X \to T$ and $Y \to T$, where T is the terminal object, and that the equaliser of $f, g \colon X \to Y$ is the pullback of the maps $\binom{f}{g} \colon X \to Y \times Y$ and $\binom{\mathrm{id}_Y}{\mathrm{id}_Y} \colon Y \to Y \times Y$.

2. Let $\mathcal{C}$ be a K-linear category. Let $X_i \in \mathcal{C}$ and suppose that the coproduct $\coprod_i X_i$ and the product $\prod_i X_i$ both exist. Show that there is a natural map

$$\coprod_i X_i \to \prod_i X_i.$$

3. Let $\mathcal{C}$ be an additive category (or just a K-linear category with a zero object).

   (a) Show that the following are equivalent for a morphism $f\colon X \to Y$.

      (i) f is a monomorphism.

      (ii) $f\theta = 0$ implies $\theta = 0$ for all $\theta\colon X' \to X$.

      (iii) $\mathrm{Ker}(f) = 0$.

   Show further that if f is a kernel for some $g\colon Y \to Z$, then f is a monomorphism.

   (b) Suppose that every morphism in $\mathcal{C}$ has both a kernel and a cokernel. Given $f\colon X \to Y$ we define $\mathrm{Coim}(f)$ to be the cokernel of $\mathrm{Ker}(f) \to X$, and $\mathrm{Im}(f)$ to be the kernel of $Y \to \mathrm{Coker}(f)$.

   Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $gf = 0$, show that f factors uniquely as

$$X \twoheadrightarrow \mathrm{Coim}(f) \to \mathrm{Ker}(g) \rightarrowtail Y,$$

   and that g factors uniquely as

$$Y \twoheadrightarrow \mathrm{Coker}(f) \to \mathrm{Im}(g) \rightarrowtail Z.$$

   Show that, as a special case, every $f\colon X \to Y$ factors uniquely as

$$X \twoheadrightarrow \mathrm{Coim}(f) \xrightarrow{\bar{f}} \mathrm{Im}(f) \rightarrowtail Y.$$

   (c) Let $\mathcal{C}$ be an additive category in which every morphism has both a kernel and a cokernel (so a pre-abelian category). Show that $\mathcal{C}$ is abelian if and only if, for every $f\colon X \to Y$, the induced map $\bar{f}\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

   Hint. Suppose $\mathcal{C}$ is abelian, and factor f as $X \overset{e}{\twoheadrightarrow} A \overset{m}{\rightarrowtail} Y$ as an epi followed by a mono. Show that $\mathrm{Ker}(f) = \mathrm{Ker}(e)$, and hence that $A \cong \mathrm{Coim}(f)$.

4. Let $\mathcal{C}$ be an abelian category. Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $gf = 0$, we know from the previous exercise that there are unique maps $\bar{f}\colon \operatorname{Im}(f) \to \operatorname{Ker}(g)$ and $\bar{g}\colon \operatorname{Coker}(f) \to \operatorname{Im}(g)$.

   (a) Show that $\bar{f}$ is an isomorphism if and only if $\bar{g}$ is an isomorphism. In this case we say that the sequence is exact (at Y).

   (b) Show that $0 \to X \xrightarrow{f} Y$ is exact if and only if f is a monomorphism.

   (c) Consider an exact commutative diagram

$$\begin{array}{ccccccccc} & & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\zeta} & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & & \end{array}$$

   we wish to construct a connecting homomorphism $\delta\colon \operatorname{Ker}(\zeta) \to \operatorname{Coker}(\xi)$. Form the pullback E of $\operatorname{Ker}(\zeta) \to Z$ and g. From the lectures we know that the kernel K of $E \to \operatorname{Ker}(\zeta)$ is isomorphic to the kernel of g. Construct a map $E \to X'$, and show that the composition $K \to E \to X' \to \operatorname{Coker}(\xi)$ is zero. Deduce that there is a map $\delta\colon \operatorname{Ker}(\zeta) \to \operatorname{Coker}(\xi)$ as claimed.

   (d) Consider a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\xi} & & & & \downarrow{\scriptstyle\zeta} & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0 \end{array}$$

   Let A be the pushout of f and ξ, and B the pullback of g and ζ. Show that there is a map $\eta\colon Y \to Y'$ making the diagram commute if and only if there exists a map $\alpha\colon A \to B$ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & X' & \longrightarrow & B & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

   Show further that in this case α is an isomorphism.

To be handed in by 21st June.