Non-commutative Algebra, WS 19/20

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 1

1. Consider the quiver Q (of type $\tilde{\mathbb{D}}_4$)

$$\begin{array}{ccccc} 1 & \searrow^{a} & & & \\ 2 & \xrightarrow{b} & 4 & \xrightarrow{d} & 5 \\ 3 & \nearrow_{c} & & & \end{array}$$

Set $P := P[3]$ and compute $\operatorname{End}(P)$.

Set I to be the following representation

$$\begin{array}{ccccc} K & \searrow^{1} & & & \\ K & \xrightarrow{1} & K & \longrightarrow & 0 \\ 0 & \nearrow & & & \end{array}$$

Compute $\operatorname{End}(I)$.

Compute $\operatorname{Hom}(P, I)$, $\operatorname{Hom}(I, P)$, $\operatorname{Ext}^1(P, P)$ and $\operatorname{Ext}^1(P, I)$.

Give a projective resolution of I, of the form $0 \to P'' \to P' \to I \to 0$, and use this to compute both $\operatorname{Ext}^1(I, I)$ and $\operatorname{Ext}^1(I, P)$.

For a general element in $\operatorname{Hom}(P'', P)$, construct the pushout, so an element in $\operatorname{Ext}^1(I, P)$. By taking elements in $\operatorname{Hom}(P'', P)$ mapping to a basis in $\operatorname{Ext}^1(I, P)$ and examining the corresponding pushouts, construct as in the tutorial an explicit functor G from representations of the Kronecker quiver $\cdot \rightrightarrows \cdot$ to representations of Q which is both fully faithful and exact, and sends the simples to I and P. (There is a certain amount of choice in these constructions.)

Recall that the functor F in the tutorial was given by

$$U \underset{B}{\overset{A}{\rightrightarrows}} V \qquad \mapsto \qquad \begin{array}{ccccc} U & \searrow^{\iota_1} & & & \\ U & \xrightarrow{\iota_2} & U^2 & \xrightarrow{(A,B)} & V \\ U & \nearrow_{\iota_3} & & & \end{array}$$

where

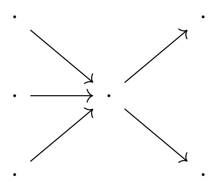
$$\iota_1 = \binom{1}{0}, \quad \iota_2 = \binom{0}{1}, \quad \iota_3 = \binom{1}{1}.$$

Given $\lambda \in K$, we have the Kronecker representation $R_1(\lambda)$ given by

$$K \underset{\lambda}{\overset{1}{\rightrightarrows}} K$$

Show that there is an automorphism σ of the set K such that $G(\lambda) \cong F(\sigma(\lambda))$ (for almost all $\lambda \in K$).

2. Consider the following quiver Γ



Construct a fully faithful exact functor from representations of the 3-Kronecker

$$\cdot \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \cdot$$

to representations of Γ.

3. The 3-Kronecker is "wild" in the sense that for every finite dimensional algebra R, there exists a finite dimensional representation X with $\mathrm{End}(X) \cong R$.

Consider the following representation of the Kronecker

$$V^n \underset{B}{\overset{A}{\rightrightarrows}} V^{n+1}$$

where

$$A = \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix}.$$

Compute the endomorphism algebra of this representation.

Now let R be any finite dimensional algebra, say with basis $x_1, \ldots, x_n$. Let $\xi_i \in \mathrm{End}_K(R)$ correspond to multiplication by x_i. Show that $R^{\mathrm{op}} \cong \mathrm{End}_R(R)$ is isomorphic to the subalgebra of $\mathrm{End}_K(R)$ given by those matrices commuting with all the ξ_i.

Finally, set X to be the representation of the 3-Kronecker

$$R^n \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; R^{n+1}$$

using matrices A and B as above, together with

$$C = \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_n \\ 0 & \cdots & 0 \end{pmatrix}$$

Compute $\mathrm{End}(X)$.

To be handed in by 28th October.