Non-commutative Algebra, WS 19/20

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Exercises 10

Throughout, A will be a finite dimensional algebra.

1. (a) For a module class $\mathcal{F}$ in A-mod, show that $\mathcal{F}$ is the torsion-free class for some torsion theory if and only if $\mathcal{F}$ is closed under extensions and submodules.

   (b) (Wakamatsu) Let $\mathcal{C}$ be a contravariantly finite module class, closed under extensions. Let $g\colon C \to M$ be a $\mathcal{C}$-cover of M, so right minimal right $\mathcal{C}$-approximation. Prove that $\mathrm{Ext}^1_A(\mathcal{C}, \mathrm{Ker}(g)) = 0$.

   Hint. Given $0 \to \mathrm{Ker}(g) \to E \to C' \to 0$ with $C' \in \mathcal{C}$, take the pushout.

2. Let t be a subfunctor of the identity on A-mod, so $t(X) \subset X$ for all modules X. We call t an idempotent radical provided $t^2(X) = t(X)$ and $t(X/t(X)) = 0$. Show that a module class $\mathcal{T}$ is the torsion class for some torsion theory if and only if $\mathcal{T} = \{t(X) : X \in A\text{-mod}\}$ for some idempotent radical t.

3. Let M be an A-module, and $B := \mathrm{End}_A(M)$.

   (a) Suppose $\mathrm{Hom}_B(Y, M) = 0$. Show that there is a natural isomorphism
   $$\mathrm{Hom}_A(L, \mathrm{Ext}^1_B(Y, M)) \cong \mathrm{Ext}^1_B(Y, \mathrm{Hom}_A(L, M))$$
   for all L with $\mathrm{Ext}^1_A(L, M) = 0$.

   Hint: consider $0 \to R \to Q \to Y \to 0$ with $Q \in \mathrm{proj}\, B$.

   (b) Consider a short exact sequence $0 \to L \to P \to X \to 0$ with
   $$\mathrm{Ext}^1_A(P \oplus L, M) = 0 = \mathrm{Hom}_A(X, M).$$
   For $\mathrm{Hom}_B(Y, M) = 0$, show that there is a natural isomorphism
   $$\mathrm{Hom}_B(Y, \mathrm{Ext}^1_A(X, M)) \cong \mathrm{Hom}_A(X, \mathrm{Ext}^1_B(Y, M)).$$

   (c) For ${}_AM$ cotilting, deduce that the functors $\mathrm{Ext}^1_A(-, M)$ and $\mathrm{Ext}^1_B(-, M)$ induce an adjoint equivalence between the categories ${}^{\perp_0}_{A}M$ and ${}^{\perp_0}_{B}M$.

4. Consider the algebra A given via

$$\text{quiver}\quad \begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, "c"] & \\ 2 \arrow[dr, "b"'] & & 3 \arrow[dl, "d"] \\ & 4 & \end{tikzcd} \qquad \text{relations} \quad ba,\ dc.$$

(a) Knit the AR quiver of A.

(b) Show that we can write A-mod as $\mathcal{X} \amalg \mathcal{S} \amalg \mathcal{Y}$ as a disjoint union of three module classes, such that

$$\mathcal{X} \amalg \mathcal{S} = \{M : \mathrm{p.dim}\, M \leq 1\} \quad \text{and} \quad \mathcal{S} \amalg \mathcal{Y} = \{M : \mathrm{i.dim}\, M \leq 1\}.$$

(c) Find all cotilting modules M; that is, modules M with $\mathrm{i.dim}\, M \leq 1$, $\mathrm{Ext}^1_A(M, M) = 0$, $\#M = 4$.

(d) For each cotilting module M compute $\mathcal{F}_M := \mathrm{cogen}\, M$. Describe the poset given by the $\mathcal{F}_M$ with respect to inclusion.

(e) Find a cotilting module M such that $\mathrm{End}_A(M)$ is hereditary, so the path algebra of a quiver. Is $\mathrm{End}_A(M)$ finite representation type?

To be handed in by 14th January.