

Non-commutative Algebra, WS 19/20

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Exercises: A. Hubery

Exercises 10

Throughout, A will be a finite dimensional algebra.

1. (a) For a module class $\mathcal{F}$ in $A\text{-mod}$, show that $\mathcal{F}$ is the torsion-free class for some torsion theory if and only if $\mathcal{F}$ is closed under extensions and submodules.
(b) (Wakamatsu) Let $\mathcal{C}$ be a contravariantly finite module class, closed under extensions. Let $g: C \rightarrow M$ be a $\mathcal{C}$ -cover of M , so right minimal right $\mathcal{C}$ -approximation. Prove that $\text{Ext}_A^1(\mathcal{C}, \text{Ker}(g)) = 0$.
Hint. Given $0 \rightarrow \text{Ker}(g) \rightarrow E \rightarrow C' \rightarrow 0$ with $C' \in \mathcal{C}$, take the pushout.
2. Let t be a subfunctor of the identity on $A\text{-mod}$, so $t(X) \subset X$ for all modules X . We call t an idempotent radical provided $t^2(X) = t(X)$ and $t(X/t(X)) = 0$. Show that a module class $\mathcal{T}$ is the torsion class for some torsion theory if and only if $\mathcal{T} = \{t(X) : X \in A\text{-mod}\}$ for some idempotent radical t .
3. Let M be an A -module, and $B := \text{End}_A(M)$.

- (a) Suppose $\text{Hom}_B(Y, M) = 0$. Show that there is a natural isomorphism

$$\text{Hom}_A(L, \text{Ext}_B^1(Y, M)) \cong \text{Ext}_B^1(Y, \text{Hom}_A(L, M))$$

for all L with $\text{Ext}_A^1(L, M) = 0$.

Hint: consider $0 \rightarrow R \rightarrow Q \rightarrow Y \rightarrow 0$ with $Q \in \text{proj } B$.

- (b) Consider a short exact sequence $0 \rightarrow L \rightarrow P \rightarrow X \rightarrow 0$ with

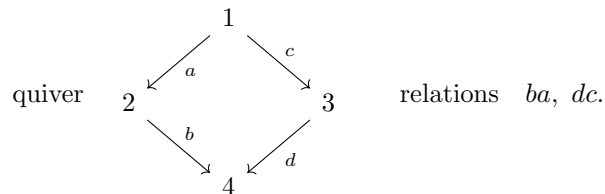
$$\text{Ext}_A^1(P \oplus L, M) = 0 = \text{Hom}_A(X, M).$$

For $\text{Hom}_B(Y, M) = 0$, show that there is a natural isomorphism

$$\text{Hom}_B(Y, \text{Ext}_A^1(X, M)) \cong \text{Hom}_A(X, \text{Ext}_B^1(Y, M)).$$

- (c) For ${}_A M$ cotilting, deduce that the functors $\text{Ext}_A^1(-, M)$ and $\text{Ext}_B^1(-, M)$ induce an adjoint equivalence between the categories ${}^{\perp_0} M$ and ${}^{\perp_0} M$.

4. Consider the algebra A given via



- (a) Knit the AR quiver of A .
- (b) Show that we can write $A\text{-mod}$ as $\mathcal{X} \amalg \mathcal{S} \amalg \mathcal{Y}$ as a disjoint union of three module classes, such that

$$\mathcal{X} \amalg \mathcal{S} = \{M : \text{p.dim } M \leq 1\} \quad \text{and} \quad \mathcal{S} \amalg \mathcal{Y} = \{M : \text{i.dim } M \leq 1\}.$$

- (c) Find all cotilting modules M ; that is, modules M with $\text{i.dim } M \leq 1$, $\text{Ext}_A^1(M, M) = 0$, $\#M = 4$.
- (d) For each cotilting module M compute $\mathcal{F}_M := \text{cogen } M$. Describe the poset given by the $\mathcal{F}_M$ with respect to inclusion.
- (e) Find a cotilting module M such that $\text{End}_A(M)$ is hereditary, so the path algebra of a quiver. Is $\text{End}_A(M)$ finite representation type?

To be handed in by 14th January.