Non-commutative Algebra, WS 19/20

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 2

1. (a) Let R be a ring and M an R-module. Show that $\operatorname{soc}(M)$ equals the intersection N of all essential submodules of M.

   Hint. To see that N is semisimple, show that every submodule $U \leq N$ has a complement. Do this by applying Zorn's Lemma to the set $\{V \leq M : U \cap V = 0\}$ to obtain an essential submodule $U \oplus V$ of M.

   (b) Consider the abelian group $\mathbb{Q}$. Show that every non-zero submodule is essential, but that $\operatorname{soc}(\mathbb{Q}) = 0$ is not essential.

2. Let R be a ring and M an R-module. We define $\operatorname{soc}^n(M)$ inductively as the preimage in M of the socle of $M/\operatorname{soc}^{n-1}(M)$. We set

$$\mathcal{A}_n := \{M : \operatorname{soc}^n(M) = M\}.$$

   Dually, we define $\operatorname{rad}^n(M)$ inductively as the radical of $\operatorname{rad}^{n-1}(M)$.

   (a) If $f\colon M \to N$ is a map of R-modules, then $f(\operatorname{soc}^n(M)) \subseteq \operatorname{soc}^n(N)$. Deduce that soc^n determines a functor $R\text{–Mod} \to \mathcal{A}_n$, which is right adjoint to the inclusion $\mathcal{A}_n \subseteq R\text{–Mod}$.

   (b) Show that if $U \leq M$ is a submodule, then $\operatorname{soc}^n(U) = U \cap \operatorname{soc}^n(M)$ for all n.

   (c) Show that if $\operatorname{soc}^n(M) = M$, then $\operatorname{rad}^n(M) = 0$.

   (d) Now suppose that $R/J(R)$ is a semisimple ring, so that every $R/J(R)$-module is semisimple. Show that $\operatorname{rad}^n(M) = J(R)^n M$, that extension of scalars identifies $R/J(R)^n\text{–Mod}$ with $\mathcal{A}_n$, and that $M \mapsto M/\operatorname{rad}^n(M)$ is left adjoint to the inclusion $\mathcal{A}_n \subseteq R\text{–Mod}$.

   (e) What goes wrong when $R/J(R)$ is not semisimple?

3. Let A be a finite dimensional algebra over a field k. Write $D = \operatorname{Hom}_k(-, k)$ for the vector space duality.

   Show that A is Frobenius if and only if $\operatorname{soc}(A)$ is isomorphic to $\operatorname{top}(A) = A/J(A)$ (as left A-modules).

   Hint.

   (a) Let M be a finite dimensional left A-module, and $E(M)$ its injective envelope. Show that $\operatorname{soc}(E(M)) = \operatorname{soc}(M)$, so that $E(M) = E(\operatorname{soc}(M))$.

   (b) Let N be a finite dimensional right A-module, so that $D(N)$ is a left A-module. Show that $\operatorname{soc}(D(N))$ is isomorphic to $D(\operatorname{top} N)$. Deduce that $D(A)$ is the injective envelope of $D(A/J(A))$ (as left A-modules).

   (c) Now use that every finite dimensional semisimple algebra B is Frobenius, so that $B \cong D(B)$ as left B-modules. (In fact, every finite dimensional semisimple algebra is symmetric, so that $B \cong D(B)$ as B-bimodules.)

4. (a) Let A be the algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "y_2"] & 2 \arrow[l, shift left, "x_2"] \arrow[r, shift left, "y_3"] & 3 \arrow[l, shift left, "x_3"] \arrow[r, shift left, dotted] & \cdot \arrow[l, shift left, dotted] \arrow[r, shift left, "y_n"] & n \arrow[l, shift left, "x_n"] \end{tikzcd}$$

together with the relations

$$y_2x_2 - x_3y_3, \quad y_3x_3 - x_4y_4, \quad \ldots \quad y_{n-1}x_{n-1} - x_ny_n, \quad y_nx_n.$$

Use the Diamond Lemma to show that A has a basis of paths of the form $x_{i+1}\cdots x_ry_r\cdots y_{j+1}$ for $1 \leq i,j \leq r \leq n$. (This is a path starting at vertex i, going up to vertex r, and then back down to vertex j, so includes the degenerate cases when $i = r$ or $j = r$.)

Compute $\dim A$.

Show that, for each j, there is a unique maximal path w_j starting at vertex j. Where does it end? Deduce that $\mathrm{soc}(P[j]) \cong S[1]$ for all j.

Show further that the bilinear form $e_1A \times Ae_1 \to K$ sending (p,q) to the coefficient of w_1 in pq is non-degenerate. Deduce that $P[1] \cong I[1]$, and hence that the algebra A is QF3 but not Frobenius.

(b) Let B be the preprojective algebra of type $\mathbb{D}_4$. This is given by the quiver

$$\begin{tikzcd} & & 2 \arrow[dl, shift left, "b"] \\ 1 \arrow[r, shift left, "a"] & 4 \arrow[l, shift left, "a^*"] \arrow[ur, shift left, "b^*"] \arrow[dr, shift left, "c^*"] & \\ & & 3 \arrow[ul, shift left, "c"] \end{tikzcd}$$

together with the relations

$$a^*a, \quad b^*b, \quad c^*c, \quad aa^* + bb^* + cc^*.$$

Find a basis of paths.

(Hint: Ignoring the relation c^*c one can apply the Diamond Lemma to obtain a basis of paths of the corresponding algebra $\tilde{B}$. Use this to compute a basis of paths for the two sided ideal I generated by cc^*, and use that $B \cong \tilde{B}/I$. You should obtain that $\dim B = 28$.)

Compute bases for the projectives $P[1]$ and $P[4]$. Deduce that $\mathrm{soc}(P[i]) \cong S[i]$ for all i, and hence that the algebra is Frobenius (but not symmetric).

To be handed in by 4th November.