

Non-commutative Algebra, WS 19/20

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Exercises 2

1. (a) Let R be a ring and M an R -module. Show that $\text{soc}(M)$ equals the intersection N of all essential submodules of M .
Hint. To see that N is semisimple, show that every submodule $U \leq N$ has a complement. Do this by applying Zorn's Lemma to the set $\{V \leq M : U \cap V = 0\}$ to obtain an essential submodule $U \oplus V$ of M .
(b) Consider the abelian group $\mathbb{Q}$. Show that every non-zero submodule is essential, but that $\text{soc}(\mathbb{Q}) = 0$ is not essential.
2. Let R be a ring and M an R -module. We define $\text{soc}^n(M)$ inductively as the preimage in M of the socle of $M/\text{soc}^{n-1}(M)$. We set

$$\mathcal{A}_n := \{M : \text{soc}^n(M) = M\}.$$

Dually, we define $\text{rad}^n(M)$ inductively as the radical of $\text{rad}^{n-1}(M)$.

- (a) If $f: M \rightarrow N$ is a map of R -modules, then $f(\text{soc}^n(M)) \subseteq \text{soc}^n(N)$. Deduce that soc^n determines a functor $R\text{-Mod} \rightarrow \mathcal{A}_n$, which is right adjoint to the inclusion $\mathcal{A}_n \subseteq R\text{-Mod}$.
 - (b) Show that if $U \leq M$ is a submodule, then $\text{soc}^n(U) = U \cap \text{soc}^n(M)$ for all n .
 - (c) Show that if $\text{soc}^n(M) = M$, then $\text{rad}^n(M) = 0$.
 - (d) Now suppose that $R/J(R)$ is a semisimple ring, so that every $R/J(R)$ -module is semisimple. Show that $\text{rad}^n(M) = J(R)^n M$, that extension of scalars identifies $R/J(R)^n\text{-Mod}$ with $\mathcal{A}_n$, and that $M \mapsto M/\text{rad}^n(M)$ is left adjoint to the inclusion $\mathcal{A}_n \subseteq R\text{-Mod}$.
 - (e) What goes wrong when $R/J(R)$ is not semisimple?
3. Let A be a finite dimensional algebra over a field k . Write $D = \text{Hom}_k(-, k)$ for the vector space duality.

Show that A is Frobenius if and only if $\text{soc}(A)$ is isomorphic to $\text{top}(A) = A/J(A)$ (as left A -modules).

Hint.

- (a) Let M be a finite dimensional left A -module, and $E(M)$ its injective envelope. Show that $\text{soc}(E(M)) = \text{soc}(M)$, so that $E(M) = E(\text{soc}(M))$.
- (b) Let N be a finite dimensional right A -module, so that $D(N)$ is a left A -module. Show that $\text{soc}(D(N))$ is isomorphic to $D(\text{top } N)$. Deduce that $D(A)$ is the injective envelope of $D(A/J(A))$ (as left A -modules).
- (c) Now use that every finite dimensional semisimple algebra B is Frobenius, so that $B \cong D(B)$ as left B -modules. (In fact, every finite dimensional semisimple algebra is symmetric, so that $B \cong D(B)$ as B -bimodules.)

4. (a) Let A be the algebra given by the quiver

$$1 \begin{array}{c} \xrightarrow{y_2} \\ \xleftarrow{x_2} \end{array} 2 \begin{array}{c} \xrightarrow{y_3} \\ \xleftarrow{x_3} \end{array} 3 \cdots \begin{array}{c} \xrightarrow{y_n} \\ \xleftarrow{x_n} \end{array} n$$

together with the relations

$$y_2x_2 - x_3y_3, \quad y_3x_3 - x_4y_4, \quad \dots \quad y_{n-1}x_{n-1} - x_ny_n, \quad y_nx_n.$$

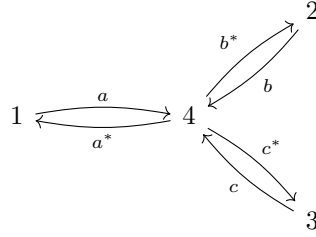
Use the Diamond Lemma to show that A has a basis of paths of the form $x_{i+1} \cdots x_r y_r \cdots y_{j+1}$ for $1 \leq i, j \leq r \leq n$. (This is a path starting at vertex i , going up to vertex r , and then back down to vertex j , so includes the degenerate cases when $i = r$ or $j = r$.)

Compute $\dim A$.

Show that, for each j , there is a unique maximal path w_j starting at vertex j . Where does it end? Deduce that $\text{soc}(P[j]) \cong S[1]$ for all j .

Show further that the bilinear form $e_1 A \times A e_1 \rightarrow K$ sending (p, q) to the coefficient of w_1 in pq is non-degenerate. Deduce that $P[1] \cong I[1]$, and hence that the algebra A is QF3 but not Frobenius.

- (b) Let B be the preprojective algebra of type $\mathbb{D}_4$. This is given by the quiver



together with the relations

$$a^*a, \quad b^*b, \quad c^*c, \quad aa^* + bb^* + cc^*.$$

Find a basis of paths.

(Hint: Ignoring the relation c^*c one can apply the Diamond Lemma to obtain a basis of paths of the corresponding algebra $\tilde{B}$. Use this to compute a basis of paths for the two sided ideal I generated by cc^* , and use that $B \cong \tilde{B}/I$. You should obtain that $\dim B = 28$.)

Compute bases for the projectives $P[1]$ and $P[4]$. Deduce that $\text{soc}(P[i]) \cong S[i]$ for all i , and hence that the algebra is Frobenius (but not symmetric).

To be handed in by 4th November.