

Non-commutative Algebra, WS 19/20

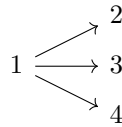
Lectures: W. Crawley-Boevey

Exercises: A. Hubery

Exercises 3

- Recall that we can compute the Auslander-Reiten translate τ of a module M by taking a (minimal) projective presentation $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$, and then taking the kernel of $\nu(f)$, where $\nu = D \operatorname{Hom}_A(-, A)$ is the Nakayama functor, so $0 \rightarrow \tau M \rightarrow \nu(P_1) \xrightarrow{\nu(f)} \nu(P_0)$.

- Let A be the path algebra of the quiver $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$ (of type $\mathbb{A}_n$). Compute $\tau(S[i])$ for each simple $S[i]$.
- Let B be the path algebra of the quiver



Compute $\tau^r(S[1])$ for all r .

- Let A be the preprojective algebra of type $\mathbb{A}_2$, so given by the quiver and relations

$$1 \xrightleftharpoons[b]{a} 2 \xrightleftharpoons[c]{d} 3 \quad ab - cd, \quad ba, \quad dc.$$

Compute $\tau^r(S[i])$ for all i and r .

- A (non-commutative) discrete valuation ring is a non-artinian ring Γ with Jacobson radical $\mathfrak{m}$ satisfying
 - every non-zero left ideal is a power of $\mathfrak{m}$, as is every non-zero right ideal, and
 - every left ideal is principal, as is every right ideal, and Γ is a local domain.

(In fact, these two conditions are equivalent.)

- Let Γ be a DVR. Show that $\Gamma\pi^n = \mathfrak{m}^n = \pi^n\Gamma$ for any $\pi \in \mathfrak{m} - \mathfrak{m}^2$, and that $\bigcap_n \mathfrak{m}^n = 0$.

Now let $H := H_r(\Gamma)$ be the following subring of $\mathbb{M}_r(\Gamma)$

$$H_r(\Gamma) := \begin{pmatrix} \Gamma & \Gamma & \cdots & \Gamma \\ \mathfrak{m} & \Gamma & \cdots & \Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{m} & \mathfrak{m} & \cdots & \Gamma \end{pmatrix} = \{(a_{ij}) \in \mathbb{M}_r(\Gamma) \mid a_{ij} \in \mathfrak{m} \text{ for all } i > j\}.$$

- (b) Show that each submodule of the indecomposable projective $P[i] = HE_{ii}$ is of the form $(\mathfrak{m}^{a_1}, \dots, \mathfrak{m}^{a_r})^t$ for some sequence of integers $a_1 \leq \dots \leq a_r \leq a_1 + 1$ with $a_{i+1} \geq 1$. Deduce that each such submodule is again indecomposable projective.
- (c) Compute $J(H)$ and $J(H)^2$.
- (d) Prove that $H/J(H)^2$ is a Nakayama algebra (each indecomposable projective left module is uniserial, as is each indecomposable projective right module). Deduce that $H/J(H)^n$ is Nakayama for all n . Deduce that the category of finite length H -modules is a uniserial category (every indecomposable finite length module is uniserial).
- (e) Let $I \leq H$ be any left ideal, so that $I \leq \bigoplus_i IE_{ii}$. By considering the projections $I \rightarrow IE_{ii}$, show that every left ideal is finitely generated projective.
- This shows that H is left hereditary and left noetherian. Dually it is right hereditary and right noetherian.
- (Conversely, it is a theorem of Michler that every basic semiperfect, hereditary noetherian prime ring is isomorphic to $H_r(\Gamma)$ for some DVR Γ and some r .)
4. (a) Let $f: X \rightarrow Y$ be a source map. Show that X is indecomposable.
- (b) Show that every irreducible map is either mono or epi.
- (c) Let $f: X \rightarrow Y$ be an irreducible epi. Show that $\text{Ker}(f)$ is indecomposable.
- (d) Show that every source map is irreducible.
- (e) Let Z be indecomposable and not projective. Show that τZ is indecomposable.
- (f) Let $f: X \rightarrow Y$ be a mono source map, giving short exact sequence

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

with Z indecomposable. Show that $X \cong \tau Z$ and that this sequence is an Auslander-Reiten sequence.

Hint. Compare the sequence to the Auslander-Reiten sequence ending at Z , which we know exists.

To be handed in by 11th November.