Non-commutative Algebra, WS 19/20

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Exercises 3

1. Recall that we can compute the Auslander-Reiten translate τ of a module M by taking a (minimal) projective presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$, and then taking the kernel of $\nu(f)$, where $\nu = D\operatorname{Hom}_A(-, A)$ is the Nakayama functor, so $0 \to \tau M \to \nu(P_1) \xrightarrow{\nu(f)} \nu(P_0)$.

   (a) Let A be the path algebra of the quiver $1 \to 2 \to \cdots \to n$ (of type $\mathbb{A}_n$). Compute $\tau(S[i])$ for each simple $S[i]$.

   (b) Let B be the path algebra of the quiver

   $$\begin{tikzcd} & 2 \\ 1 \arrow[ur] \arrow[r] \arrow[dr] & 3 \\ & 4 \end{tikzcd}$$

   Compute $\tau^r(S[1])$ for all r.

2. Let A be the preprojective algebra of type $\mathbb{A}_2$, so given by the quiver and relations

   $$1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \underset{c}{\overset{d}{\rightleftarrows}} 3 \qquad ab - cd,\ ba,\ dc.$$

   Compute $\tau^r(S[i])$ for all i and r.

3. A (non-commutative) discrete valuation ring is a non-artinian ring Γ with Jacobson radical $\mathfrak{m}$ satisfying

   - every non-zero left ideal is a power of $\mathfrak{m}$, as is every non-zero right ideal, and
   - every left ideal is principal, as is every right right ideal, and Γ is a local domain.

   (In fact, these two conditions are equivalent.)

   (a) Let Γ be a DVR. Show that $\Gamma\pi^n = \mathfrak{m}^n = \pi^n\Gamma$ for any $\pi \in \mathfrak{m} - \mathfrak{m}^2$, and that $\bigcap_n \mathfrak{m}^n = 0$.
   Now let $H := H_r(\Gamma)$ be the following subring of $\mathbb{M}_r(\Gamma)$

   $$H_r(\Gamma) := \begin{pmatrix} \Gamma & \Gamma & \cdots & \Gamma \\ \mathfrak{m} & \Gamma & \cdots & \Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{m} & \mathfrak{m} & \cdots & \Gamma \end{pmatrix} = \{(a_{ij}) \in \mathbb{M}_r(\Gamma) \mid a_{ij} \in \mathfrak{m} \text{ for all } i > j\}.$$

(b) Show that each submodule of the indecomposable projective $P[i] = HE_{ii}$ is of the form $(\mathfrak{m}^{a_1}, \dots, \mathfrak{m}^{a_r})^t$ for some sequence of integers $a_1 \leq \cdots \leq a_r \leq a_1 + 1$ with $a_{i+1} \geq 1$. Deduce that each such submodule is again indecomposable projective.

(c) Compute $J(H)$ and $J(H)^2$.

(d) Prove that $H/J(H)^2$ is a Nakayama algebra (each indecomposable projective left module is uniserial, as is each indecomposable projective right module). Deduce that $H/J(H)^n$ is Nakayama for all n. Deduce that the category of finite length H-modules is a uniserial category (every indecomposable finite length module is uniserial).

(e) Let $I \leq H$ be any left ideal, so that $I \leq \bigoplus_i IE_{ii}$. By considering the projections $I \twoheadrightarrow IE_{ii}$, show that every left ideal is finitely generated projective.

This shows that H is left hereditary and left noetherian. Dually it is right hereditary and right noetherian.

(Conversely, it is a theorem of Michler that every basic semiperfect, hereditary noetherian prime ring is isomorphic to $H_r(\Gamma)$ for some DVR Γ and some r.)

4. (a) Let $f \colon X \to Y$ be a source map. Show that X is indecomposable.

(b) Show that every irreducible map is either mono or epi.

(c) Let $f \colon X \to Y$ be an irreducible epi. Show that $\mathrm{Ker}(f)$ is indecomposable.

(d) Show that every source map is irreducible.

(e) Let Z be indecomposable and not projective. Show that τZ is indecomposable.

(f) Let $f \colon X \to Y$ be a mono source map, giving short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

with Z indecomposable. Show that $X \cong \tau Z$ and that this sequence is an Auslander-Reiten sequence.

Hint. Compare the sequence to the Auslander-Reiten sequence ending at Z, which we know exists.

To be handed in by 11th November.