Non-commutative Algebra, WS 19/20

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 5

1. Some more practice with injectives.

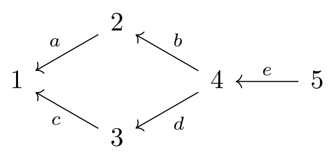
   (a) Let $f\colon V \to W$ be a linear map between finite dimensional vector spaces, and $D(f)\colon D(W) \to D(V)$ its dual. Show that, if f is represented by a matrix M with respect to a choice of bases for V and W, then $D(f)$ is represented by the transpose M^t with respect to the dual bases of $D(W)$ and $D(V)$.

   (b) Let Q be a quiver, and Q^{op} its opposite quiver, so Q^{op} has an arrow $\bar{a}$ for each arrow $a \in Q$. Show that $a \mapsto \bar{a}$ extends to an algebra isomorphism $(kQ)^{\mathrm{op}} \xrightarrow{\sim} kQ^{\mathrm{op}}$.

   More generally, show that if $A = kQ/I$, then $A^{\mathrm{op}} \xrightarrow{\sim} kQ^{\mathrm{op}}/\bar{I}$.

   (c) Let $A = kQ/I$ be finite dimensional, and set $\bar{A} := A^{\mathrm{op}}$. We have the projective left $\bar{A}$-module $\bar{P}[i] := \bar{A}\bar{e}_i$, which we can draw as a representation of Q^{op} using matrices $M_{\bar{a}}$ indexed by the arrows $\bar{a} \in Q^{\mathrm{op}}$. Show that $D(\bar{P}[i]) = I[i]$ is the corresponding injective left A-module, which we can draw as a representation of Q using the matrices $M_a := M_{\bar{a}}^t$.

   (d) Apply this construction to write down the indecomposable injective $I[1]$ for the algebra A given by

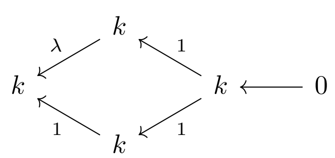
   quiver 

   with relation $(ab - cd)e$.

   (e) Given an element $x \in e_i A e_j$, we obtain a homomorphism $f_x\colon P[i] \to P[j]$. Also, $\bar{x} \in \bar{e}_j \bar{A} \bar{e}_i$, so we have a homomorphism $\bar{f}_{\bar{x}}\colon \bar{P}[j] \to \bar{P}[i]$, and thus a homomorphism $D(\bar{f}_{\bar{x}})\colon I[i] \to I[j]$.

   Show that, under the canonical isomorphisms $\mathrm{Hom}_A(P[i], A) \cong \bar{P}[i]$ of left $\bar{A}$-modules, the map $\mathrm{Hom}_A(f_x, A)$ corresponds to $\bar{f}_{\bar{x}}$, and hence that the Nakayama functor acts as $\nu(f_x) = D(\bar{f}_{\bar{x}})$.

   (f) Take the algebra A as in (d), and apply this to compute $\tau(M)$ for the A-module M given by

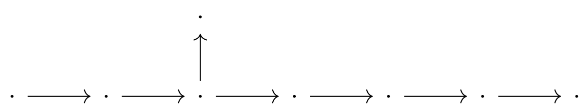
   

   $\lambda \in k$.

2. Write out a full proof, giving all the details, of the theorem in Section 1.6 showing that every indecomposable module over a Nakayama algebra is uniserial.

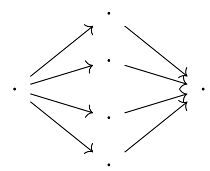
   Give an example of a Nakayama algebra A and indecomposable projective-injective P with socle S such that (i) $SA = S$ (ii) $SA \neq S$.

3. Consider the following quiver Q (of type $\mathbb{E}_8$)

   and let I be the ideal in kQ generated by all paths of length 3. Knit the Auslander-Reiten quiver of kQ/I.

4. Consider the following quiver

   with relations such that $r = p + \lambda q$ and $s = p + \mu q$, where p, q, r, s are the four paths from left to right, and $\lambda, \mu \in k$.

   Explain why we can knit a preprojective component $\mathcal{C}$ of the Auslander-Reiten quiver containing all but one of the indecomposable projective modules.

To be handed in by 25th November.