Non-commutative Algebra, WS 19/20

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Exercises 7

1. Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be contravariant additive functors between additive categories. We say that these form an adjoint pair provided we have natural transformations

$$\theta_X : X \to GF(X) \text{ for } X \in \mathcal{A} \quad \text{and} \quad \eta_Y : Y \to FG(Y) \text{ for } Y \in \mathcal{B}$$

yielding an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(X, G(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(Y, F(X)), \quad f \mapsto F(f)\eta_Y \quad \text{for all } X \in \mathcal{A},\ Y \in \mathcal{B}$$

with inverse $g \mapsto G(g)\theta_X$.

(a) Let $f\colon X \to X'$ be a map in $\mathcal{A}$. Show that $GF(f)\theta_X = \theta_{X'}f$.

(b) Deduce that the composite

$$\mathrm{Hom}_{\mathcal{A}}(X, X') \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(F(X'), F(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X, GF(X'))$$

is given by $f \mapsto \theta_{X'}f$.

(c) Show that F is (fully) faithful if and only if each θ_X is a monomorphism (isomorphism).

(d) Prove that F and G induce an antiequivalence between the full subcategories $\{X : \theta_X \text{ is an iso}\}$ and $\{Y : \eta_Y \text{ is an iso}\}$.

2. Let A and B be finite dimensional K-algebras, and M a finite dimensional left $A \otimes_K B$-module; equivalently M is both a left A-module and a left B-module, and these actions commute, so $a(bm) = b(am)$ for all $a \in A$, $b \in B$ and $m \in M$.

(a) Show that the contravariant functors $F := \mathrm{Hom}_A(-, M)$ and $\mathrm{Hom}_B(-, M)$ together with the evaluation maps

$$\theta_X := \mathrm{ev}_X\colon X \to \mathrm{Hom}_B(\mathrm{Hom}_A(X, M), M)$$

and

$$\eta_Y := \mathrm{ev}_Y\colon Y \to \mathrm{Hom}_A(\mathrm{Hom}_B(Y, M), M)$$

form an adjoint pair between A-mod and B-mod.
Note that θ_A is the natural map $A \to \mathrm{End}_B(M)$.

(b) Show that F is faithful if and only if ${}_AM$ is a cogenerator (which implies that ${}_AM$ is faithful).
Hint: Given $X \in A$-mod, take a left add(M)-approximation $X \to M'$, say with kernel K. Show that for $f\colon X' \to X$ we have $F(f) = 0$ if and only if $\mathrm{Im}(f) \subset K$.

(c) Assume that ${}_AM$ is a cogenerator. Show that F is fully faithful if and only if θ_M is an isomorphism.

Hint: Take a left approximation $X \rightarrowtail M'$, say with cokernel C. Show that the functor GF sends this to a left exact sequence. Compare with the original sequence using θ.

You may find the following result useful. Let $\phi\colon F_1 \Rightarrow F_2$ be a natural transformation between additive functors. If ϕ_X is an isomorphism, then $\phi_{X'}$ is an isomorphism for all $X' \in \mathrm{add}(X)$. See for example Exercise 8.3 from last semester.

3. We keep the setting as in Q2. Assume further that $B := \mathrm{End}_A(M)$.

   (a) Show that θ_M is an isomorphism. (In particular, if ${}_AM$ is a cogenerator, then F is fully faithful and M is faithfully balanced.)

   (b) Show that F and G restrict to an antiequivalence between $\mathrm{add}({}_AM)$ and $\mathrm{proj}(B)$.

   Hint: You may find the following result useful. Let $\phi\colon F_1 \Rightarrow F_2$ be a natural transformation between additive functors. If ϕ_X is an isomorphism, then $\phi_{X'}$ is an isomorphism for all $X' \in \mathrm{add}(X)$. See for example Exercise 8.3 from last semester.

   (c) Write out a proof that $X \in \mathrm{cogen}_0 M$ if and only if θ_X is a monomorphism.

   (d) Write out a proof that $X \in \mathrm{cogen}_1 M$ if and only if θ_X is an isomorphism.

   (e) Suppose $X \in \mathrm{cogen}_0 M$, and take any left $\mathrm{add}(M)$-approximation $f\colon X \rightarrowtail M'$. Prove that $X \in \mathrm{cogen}_1 M$ if and only if $\mathrm{Coker}(f) \in \mathrm{cogen}_0 M$.

   (f) Let $n \geq 2$. Write out a proof that $X \in \mathrm{cogen}_n(M)$ if and only if θ_X is an isomorphism and $\mathrm{Ext}^i_B(\mathrm{Hom}_A(X,M),M) = 0$ for all $0 < i < n$.

4. We keep the setting as in Q2. Let $D = \mathrm{Hom}_K(-,K)$ be the vector space duality, yielding a contravariant functor $A^{\mathrm{op}}\text{-mod} \to A\text{-mod}$. Composing D with $F = \mathrm{Hom}_A(-,M)$ and $G = \mathrm{Hom}_B(-,M)$ yields the covariant functors

$$FD\colon A^{\mathrm{op}}\text{-mod} \to B\text{-mod} \quad \text{and} \quad DG\colon B\text{-mod} \to A^{\mathrm{op}}\text{-mod}.$$

   (a) Show that $\mathrm{Hom}_A(X, DX') \cong D(X' \otimes_A X) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(X', DX)$.

   (b) Show that we have natural isomorphisms

$$FD(X) = \mathrm{Hom}_A(DX, M) \xrightarrow{\sim} \mathrm{Hom}_{A^{\mathrm{op}}}(DM, X), \quad f \mapsto \mathrm{ev}_X^{-1} \circ Df$$

   and

$$DM \otimes_B Y \xrightarrow{\sim} D\,\mathrm{Hom}_B(Y,M) = DG(Y), \quad \mu \otimes y \mapsto \big(h \mapsto \mu(h(y))\big).$$

   (c) Show that DG is left adjoint to FD, via the unit and counit

$$F(\mathrm{ev}_{GY})^{-1} \circ \eta_Y \colon Y \mapsto FD^2GY \quad \text{and} \quad \mathrm{ev}_X \circ \varepsilon_X \colon FD^2GX \to X.$$

   In particular, D restricts to an antiequivalence between $\mathrm{gen}_1(DM)$ in A^{op}-mod and $\mathrm{cogen}_1 M$ in A-mod.

To be handed in by 9th December.