Non-commutative Algebra, WS 19/20

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 8

1. Let A be a finite dimensional K-algebra. We want to show that the assignment $\mathbb{Y}\colon X \mapsto \mathrm{Hom}_A(-, X)$ induces an equivalence of categories between A-mod and the category $\mathrm{Fun}((\mathrm{proj}\, A)^{\mathrm{op}}, K\text{-mod})$ of contravariant additive functors from $\mathrm{proj}\, A$ to the category K-mod of finite dimensional K-vector spaces.

   (a) Show that $\mathbb{Y}$ determines a fully faithful functor

   $$A\text{-mod} \to \mathrm{Fun}((\mathrm{proj}\, A)^{\mathrm{op}}, K\text{-mod}).$$

   (b) Let $h \in \mathrm{Fun}((\mathrm{proj}\, A)^{\mathrm{op}}, K\text{-mod})$. Show that there is a K-algebra homomorphism

   $$A \xrightarrow{\sim} \mathrm{End}_A(A)^{\mathrm{op}} \to \mathrm{End}_K(h(A)), \quad a \mapsto h(\rho_a),$$

   where $\rho_a \in \mathrm{End}_A(A)$ is right multiplication by a. This endows $h(A)$ with the structure of a left A-module.
   Show further that this extends to a functor

   $$\mathrm{ev}_A\colon \mathrm{Fun}((\mathrm{proj}\, A)^{\mathrm{op}}, K\text{-mod}) \to A\text{-mod}.$$

   (c) By Yoneda's Lemma we have an isomorphism between elements of $h(P)$ and natural transformations $\mathrm{Hom}_A(-, P) \Rightarrow h$, say sending $x \in h(P)$ to η_x.
   Show that the assignment $h(P) \to \mathrm{Hom}_A(P, h(A))$ sending x to the homomorphism $P \xrightarrow{\sim} \mathrm{Hom}_A(A, P) \xrightarrow{\eta_{x,A}} h(A)$ yields a natural transformation $h \Rightarrow \mathrm{Hom}_A(-, h(A))$.
   Show further that this is an isomorphism at A, so is an isomorphism on all of $\mathrm{proj}\, A$, using the same hint as from 3(b).
   This finishes the proof.

Let Q be a quiver. To define a contravariant functor $h\colon \operatorname{proj} KQ \to K\text{-mod}$, we first need to specify a vector space $h_i := h(P[i])$ for each vertex i of Q. Next, $\operatorname{Hom}_{KQ}(P[j], P[i])$ has basis the set of paths from i to j, each of which is a product of arrows. Since h preserves composition of homomorphisms, it is enough to specify a linear map $h_a\colon h_i \to h_j$ for each arrow $a\colon i \to j$. Conversely, every such choice determines a functor. Thus contravariant functors $\operatorname{proj} KQ \to K\text{-mod}$ are the same as K-representations of Q, which we know is equivalent to $KQ\text{-mod}$.
In general, let $A = KQ/I$ be a quotient of a path algebra, and $h\colon \operatorname{proj} KQ \to K\text{-mod}$ a contravariant functor. Now each element $p \in I$ determines a homomorphism of projective KQ-modules, and it follows that h determines a functor $\operatorname{proj} A \to K\text{-mod}$ if and only if $h(p) = 0$ for all $p \in I$. In other words, we can identify $A\text{-mod}$ with the subcategory of those K-representations of Q satisfying the relations I.

2. A faithfully balanced pair (A, M) consists of a finite dimensional algebra A and a faithfully balanced A-module M. Its endomorphism correspondent is the pair (B, M), where $B = \operatorname{End}_A(M)$. Note that ${}_BM$ will necessarily be faithfully balanced.

   Two faithfully balanced pairs (A, M) and (A', M') are equivalent provided there exists an equivalence $\alpha\colon A\text{-mod} \xrightarrow{\sim} A'\text{-mod}$ restricting to an equivalence $\operatorname{add} M \cong \operatorname{add} M'$.

   We wish to show that equivalent faithfully balanced pairs have equivalent endomorphism correspondents.

   (a) Show that $F := \operatorname{Hom}_A(-, M)$ and $G := \operatorname{Hom}_B(-, M)$ induce antiequivalences between $\operatorname{proj} A$ and $\operatorname{add} {}_BM$, and between $\operatorname{proj} B$ and $\operatorname{add} {}_AM$. In particular, the map θ_X is an isomorphism for all $X \in \operatorname{add}(A \oplus M)$, and η_Y is an isomorphism for all $Y \in \operatorname{add}(B \oplus M)$.

   (b) Let $\alpha'\colon A'\text{-mod} \xrightarrow{\sim} A\text{-mod}$ be an adjoint to α. We know that α induces an equivalence $\beta := F'\alpha G\colon \operatorname{proj} B \xrightarrow{\sim} \operatorname{proj} B'$, with adjoint $\beta' := F\alpha' G'$. Show that β extends to an equivalence

   $$B\text{-mod} \cong \operatorname{Fun}((B\text{-mod})^{\mathrm{op}}, K\text{-mod}) \cong \operatorname{Fun})(B'\text{-mod})^{\mathrm{op}}, K\text{-mod}) \cong B'\text{-mod}$$

   sending a B-module Y to the B'-module $\operatorname{Hom}_B(F\alpha'(M'), Y)$. Note that the action of B' is via the map $B' = \operatorname{End}_{A'}(M') \to \operatorname{End}_A(\alpha'(M'))$ induced by α'.

   (c) Show that the B-module ${}_BM = F(A)$ is sent to $F'(\alpha(A)) \in \operatorname{add} {}_{B'}M'$.

   (d) Deduce that β restricts to an equivalence $\operatorname{add} {}_BM \cong \operatorname{add} {}_{B'}M'$.

3. Let K be an algebraically closed field, and A a finite dimensional K-algebra. Set $B := \mathrm{End}_A(M)$ for some $M \in A$-mod. Assume that $M = M_1 \oplus \cdots \oplus M_n$ with $\mathrm{End}_A(M_i) = K$ for all i, and $M_i \not\cong M_j$ for all $i \neq j$. Recall that $\mathrm{Hom}_A(-, M)$ induces an equivalence $\operatorname{add} M \cong (\operatorname{proj} B)^{\mathrm{op}}$.

   (a) Show that $B \cong K\Gamma/J$ for some finite quiver Γ having n-vertices and some admissible ideal J. Show further that $J(B) = \bigoplus_{i \neq j} \mathrm{Hom}_A(M_i, M_j)$.

   (b) Show that the module ${}_BM$, when drawn as a quiver representation for Γ, has vector space M_j at vertex j. Moreover, if an arrow $a\colon i \to j$ in Γ is represented by a homomorphism $\hat{a} \in \mathrm{Hom}_A(M_i, M_j)$, then the linear map at a is precisely $\hat{a}$.

   (c) Let $A = KQ$ for the quiver $Q\colon 1 \to 2 \to 3$. This has Auslander-Reiten quiver

$$\Gamma_A\colon \quad \begin{tikzcd}[column sep=small] & & P[1] \arrow[dr] & & \\ & P[2] \arrow[ur] \arrow[dr] & & I[2] \arrow[dr] & \\ S[1] \arrow[ur] & & S[2] \arrow[ur] & & S[3] \end{tikzcd}$$

   For the module $M = S[1] \oplus P[2] \oplus P[1] \oplus I[2]$ compute $B := \mathrm{End}_A(M) = K\Gamma/J$ and draw ${}_BM$ as a quiver representation

   (d) Do the same for the module $M' = P[1] \oplus S[2] \oplus I[2] \oplus S[3]$, having endomorphism algebra $B' = K\Gamma'/J'$.

4. Let Q and Γ be two finite quivers. We define their Cartesian product $Q \times \Gamma$ to be the quiver with vertices $Q_0 \times \Gamma_0$ and arrows $a_p\colon (i, p) \to (j, p)$ for $a\colon i \to j$ in Q_1, as well as $b_i\colon (i, p) \to (i, q)$ for $b\colon p \to q$ in Γ_1.

   (a) Show that $KQ \otimes_K K\Gamma$ is isomorphic to $K(Q \times \Gamma)$ modulo the ideal C given by commutative squares, so $a_q b_i - b_j a_p$ for all $a\colon i \to j$ in Q_1 and $b\colon p \to q$ in Γ_1.

   Hint. Show first that there is a natural surjective algebra homomorphism $K(Q \times \Gamma) \to KQ \otimes_K K\Gamma$ (though a better approach would be to use the relevant universal properties).

   (b) If $A = KQ/I$ and $B = KQ/J$, show that $A \otimes_K B$ is isomorphic to the quotient of $KQ \otimes_K \Gamma$ modulo the ideal $I \otimes K\Gamma + KQ \otimes J$.

   (c) For the setting of Q3 (c), write the $A \otimes_K B$-module M as a quiver representation for $Q \times \Gamma$.

   (d) Do the same for the setting of Q3 (d), writing the $A \otimes_K B'$-module M' as a quiver representation for $Q \times \Gamma'$.

To be handed in by 16th December.