Non-commutative Algebra 3, SS 2020

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Exercises 1

1. Let X be a tolopological space. Recall that a subset is closed provided its complement is open.

   (a) Show that an arbitrary intersection of closed sets is again closed.

   (b) We define the closure of a subset $Y \subset X$ to be the intersection over all closed subsets of X containing Y.

   Prove that the following conditions are equivalent for a subset $Y \subset X$.

   (i) Y is an open subset of a closed subset of X.

   (ii) Y is an open subset of its closure.

   (iii) Y is the intersection of an open and a closed subset of X.

   Such a subset Y is said to be locally closed.

   (c) Now let $Y \subset X$ be a locally closed subset. Show that a subset $Z \subset Y$ is locally closed in Y if and only if it is locally closed in X.

2. Let $\theta\colon Z \to Y$ and $\phi\colon Y \to X$ be morphisms of spaces with functions. Show that the composition $\phi\theta\colon Z \to X$ is a morphism of spaces with functions.

3. Let X be a space with functions, and Y a subset of X with its induced structure as a space with functions.

   (a) Show that the inclusion $\iota\colon Y \hookrightarrow X$ is a morphism of spaces with functions.

   (b) Let Z be any space with functions and $\theta\colon Z \to Y$ any map. Show that θ is a morphism of spaces if and only if $\iota\theta\colon Z \to X$ is a morphism of spaces. In other words, morphisms $Z \to Y$ can be thought of as morphisms $Z \to X$ with image contained in Y.

> The definition of a space with functions is difficult to check: for example one needs to consider all possible open covers of all open subsets, and their corresponding functions. The next exercise can be regarded as a technical lemma which minimises the amount of data needed to define a space with functions.

4. Let X be a set, and $\mathcal{B}$ a collection of subsets of X which is closed under finite intersections and contains $\emptyset$ and X. Suppose further that we have algebras $\mathcal{O}'_X(U)$ for each $U \in \mathcal{B}$ with the property that, if $f \in \mathcal{O}'_X(U)$ and $V \subset U$ in $\mathcal{B}$, then $f|_V \in \mathcal{O}'_X(V)$.

   (a) Define the distinguished open sets as

$$D(g,U) := \{u \in U : g(u) \neq 0\} \quad \text{for} \quad U \in \mathcal{B},\ g \in \mathcal{O}'_X(U).$$

Show that this collection of subsets is again closed under finite intersections, and contains $\mathcal{B}$.

Show that there is a topology on X whose open sets are precisely the arbitrary unions of distinguished opens.

(b) Let $W \subset X$ be open. We define $\mathcal{O}_X(W)$ to be the set of those functions $h\colon W \to K$ for which there exists, for each point $w \in W$, an open $U \in \mathcal{B}$ and $f, g \in \mathcal{O}'_X(U)$ such that

- $w \in D(g, U)$.
- $h = f/g$ on $W \cap D(g, U)$.

Show that this construction gives X the structure of a space with functions.

(c) Let Z be any space with functions, and $\theta\colon Z \to X$ any map. Show that θ is a morphism of spaces with functions if and only if

- $\theta^{-1}(U)$ is open in Z for all $U \in \mathcal{B}$.
- $f\theta \in \mathcal{O}_Z(\theta^{-1}(U))$ for all $U \in \mathcal{B}$ and $f \in \mathcal{O}'_X(U)$.

5. Apply the previous result to the set K^n, using $\mathcal{B} = \{K^n, \emptyset\}$ together with $\mathcal{O}'(K^n) = K[X_1, \ldots, X_n]$ and $\mathcal{O}'(\emptyset) = 0$, where the X_i are the co-ordinate functions on K^n.

   (a) Show that the resulting space with functions is $\mathbb{A}^n$.

   (b) Deduce moreover that if Z is a space with functions and $\theta\colon Z \to \mathbb{A}^n$ is any map, then θ is a morphism of spaces with functions if and only if $X_i\theta \in \mathcal{O}(Z)$ for all i.

6. We can also apply this result to describe the product. Let X and Y be spaces with functions.

   (a) Using Exercise (4) show that we can endow the set $X \times Y$ with the structure of a space with functions as follows.

   We take $\mathcal{B}$ to be the collection of $U \times V$ such that $U \subset X$ and $V \subset Y$ are both open. We take $\mathcal{O}'_{X\times Y}(U \times V)$ to be those functions h such that

$$h(u, v) := \sum_{\text{finite}} f_i(u) g_i(v), \quad \text{with } f_i \in \mathcal{O}_X(U) \text{ and } g_i \in \mathcal{O}_Y(V).$$

   (b) Show that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are both morphisms of spaces with functions.

   (c) Let $p_X : Z \to X$ and $p_Y : Z \to Y$ be morphisms of spaces with functions. Show that there is a unique morphism of spaces with functions $p\colon Z \to X \times Y$ such that $p_X = \pi_X p$ and $p_Y = \pi_Y p$.

This proves that $X \times Y$ with the above structure as a space with functions is a categorical product. Namely, we have morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that the induced map

$$\mathrm{Hom}(Z, X \times Y) \to \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y), \quad p \mapsto (\pi_X p, \pi_Y p),$$

is bijective.

To be handed in by 4th May.