Non-commutative Algebra 3, WS 2017

Lectures: W. Crawley-Boevey
Exercises: A. Hubery

Exercises 10

1. Let $K = \mathbb{F}_q$ be a finite field with q elements. Set $\mathcal{N}_n$ to be the set of all nilpotent matrices in $\mathbb{M}_n(K)$. In this question we want to prove $|\mathcal{N}_n| = q^{n(n-1)}$.

   Recall that every matrix $M \in \mathbb{M}_n(K)$ determines a $K[T]$-module of dimension n, by letting T act as M. Then, given $M, M' \in \mathbb{M}_n(K)$, we have

   $$\mathrm{Hom}_{K[T]}(M, M') = \{\theta \in \mathbb{M}_n(K) : \theta M = M'\theta\}.$$

   Set $N = J_n(0)$ to be the Jordan block

   $$N := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

   and consider the set $\mathcal{S}_n$ of pairs (A, θ) such that $A \in \mathcal{N}_n$ and $\theta \in \mathrm{Hom}_{K[T]}(N, A)$.

   (a) Show that for every nilpotent matrix A we have $\dim \mathrm{Hom}_{K[T]}(N, A) = n$. Thus the projection $\mathcal{S}_n \to \mathcal{N}_n$ on to the first co-ordinate is surjective and every fibre has size q^n. In other words, $|\mathcal{S}_n| = q^n |\mathcal{N}_n|$.

   (b) Now consider the projection $\mathcal{S}_n \to \mathbb{M}_n(K)$ on to the second co-ordinate. Show that the fibre over θ has the same size as the fibre over $M\theta$ for every $M \in \mathrm{GL}_n(K)$. Thus we may assume that θ is in row-reduced form.

   (c) Since $\theta N = A\theta$, we know that $N(\mathrm{Ker}(\theta)) \subset \mathrm{Ker}(\theta)$. Assuming θ is in row reduced form, show that we must have $\theta = E_r := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $I_r \in \mathbb{M}_r(K)$ is the identity matrix and $r = \mathrm{rank}\,\theta$.

   (d) Show that for $\theta = E_r$, the number of nilpotent matrices A for which $\theta N = A\theta$ is $q^{r(n-r)}|\mathcal{N}_{n-r}|$. By induction this equals $q^{(n-1)(n-r)}$ for $r > 0$.

   (e) Show that the number of θ which have row reduced form E_r is $(q^n - 1)(q^n - q)\cdots(q^n - q^{r-1})$.

   (f) It follows that

   $$|\mathcal{S}_n| - |\mathcal{N}_n| = \sum_{r>0} (q^n - 1)(q^n - q)\cdots(q^n - q^{r-1})q^{(n-1)(n-r)}.$$

   Prove that this equals $(q^n - 1)q^{n(n-1)}$, and hence that $|\mathcal{N}_n| = q^{n(n-1)}$.

To be handed in by 15th January.