Non-commutative Algebra 3, WS 2017

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Exercises 11

Let Q be a quiver, with vertex set $\{1,\dots,n\}$ and no vertex loops. We then have the root lattice $\mathbb{Z}^n$, which comes equipped with the symmetric bilinear form

$$(e_i, e_j) := 2\delta_{ij} - b_{ij}.$$

Here $b_{ij} = a_{ij} + a_{ji}$ and a_{ij} is the number of arrows $i \to j$ in Q. We have a partial order on the root lattice by setting $\alpha \geq \beta$ provided $\alpha = \sum_i \alpha_i e_i$ and $\beta = \sum_i \beta_i e_i$ satisfy $\alpha_i \geq \beta_i$ for all i.

The Weyl group W is the group of automorphisms of $\mathbb{Z}^n$, generated by the simple reflections

$$s_i \colon x \mapsto x - (x, e_i)e_i.$$

The set of real roots is

$$\Phi^{\mathrm{re}} := \{w(e_i) : w \in W,\, 1 \leq i \leq n\}.$$

1. Here we prove some basic properties of the Weyl group action,

   (a) Show that the Weyl group preserves the bilinear form, so

   $$(wx, wy) = (x, y) \quad \text{for all } x, y \in \mathbb{Z}^n \text{ and all } w \in W.$$

   (b) Set $\alpha := w(e_i)$. Show that ws_iw^{-1} acts as

   $$x \mapsto x - (x, \alpha)\alpha,$$

   and hence depends only on the real root α, and not on the presentation $\alpha = w(e_i)$. We denote this element of the Weyl group by s_α and call it the reflection at α.

2. We now compute some relations in the Weyl group.

   (a) Show that $s_i^2 = 1$ for all i.

   (b) Now take $i \neq j$. Relabelling we may assume $i = 1$ and $j = 2$. Show that the matrix representing s_1s_2 is of the form

   $$\begin{pmatrix} M & N \\ 0 & I \end{pmatrix}$$

   where $M \in \mathrm{GL}_2(\mathbb{Z})$, and $I \in \mathrm{GL}_{n-2}(\mathbb{Z})$ is the identity matrix. Compute the matrix M explicitly, in terms of the number $b = b_{12}$.

   (c) Show that if $b = 0$, then $(s_1s_2)^2 = 1$.
   Show that if $b = 1$, then $(s_1s_2)^3 = 1$.
   Show that if $b \geq 2$, then s_1s_2 has infinite order.

A group $\tilde{W}$ having generators $\tilde{s}_i$ and relations $\tilde{s}_i^2 = 1$, $(\tilde{s}_i\tilde{s}_j)^{m_{ij}} = 1$ for some m_{ij} is called a Coxeter group. If we take $m_{ij} = 2, 3, \infty$ according to whether $b_{ij} = 0, 1, \geq 2$, then we see that there is a surjective group homomorphism $\tilde{W} \twoheadrightarrow W$, $\tilde{s}_i \mapsto s_i$. In fact, this is an isomorphism as we see below.

3. Let $\tilde{W}$ be the Coxeter group as above. We define the length $\ell(\tilde{w})$ of an element $\tilde{w} \in \tilde{W}$ to be the length of a shortest expression $\tilde{w} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_r}$ of $\tilde{w}$ as a product of generators. Clearly $\ell(\tilde{w}) = 0$ if and only if $\tilde{w} = 1$ and $\ell(\tilde{w}) = 1$ if and only if $\tilde{w} = \tilde{s}_i$ for some i.

   (a) Suppose $\tilde{w} \mapsto w$ under the map $\tilde{W} \to W$. Since W acts on the root lattice, we can regard w as an element of $\mathrm{GL}_n(\mathbb{Z})$. Show that $\det(w) = (-1)^{\ell(\tilde{w})}$.

   (b) Deduce that $\ell(\tilde{w}\tilde{s}_i) = \ell(\tilde{w}) \pm 1$ for all $\tilde{w}$ and all $\tilde{s}_i$.

4. A fundamental result in the theory of Weyl groups is the following, relating the length function to the action on the root lattice.

$$\ell(\tilde{w}\tilde{s}_i) > \ell(\tilde{w}) \quad \text{if and only if} \quad w(e_i) > 0.$$

   Dually we have

$$\ell(\tilde{w}\tilde{s}_i) < \ell(\tilde{w}) \quad \text{if and only if} \quad w(e_i) < 0.$$

   Use this result to prove the following three statements.

   (a) The surjective group homomorphism $\tilde{W} \twoheadrightarrow W$ described above is an isomorphism.

   (b) Every real root is either positive or negative, so $\alpha \in \Phi^{\mathrm{re}}$ implies $\alpha > 0$ or $\alpha < 0$.

   (c) Given $w \in W$, the number of real roots α satisfying $\alpha > 0 > w(\alpha)$ is precisely $\ell(w)$. (Hint: write $w = w's_i$ with $\ell(w') < \ell(w)$ and use induction on length.)

To be handed in by 22nd January.