Non-commutative Algebra 3, WS 2017

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Exercises 12

We work over an algebraically closed field K.

1. Given a group G, we define its derived subgroup G' to be the subgroup generated by all commutators $[g,h] := ghg^{-1}h^{-1}$.

   (a) Show that $G' \lhd G$ is a normal subgroup, and that G/G' is abelian.

   (b) Show that any subgroup $G' \leq H \leq G$ is necessarily normal in G.

> Iterating this, we set $G^{(0)} := G$, and $G^{(n+1)}$ to be the derived subgroup of $G^{(n)}$, so that $G^{(1)} = G'$. A group is said to be solvable if $G^{(n)}$ is trivial for some n.

2. Set $B_n \subset \mathrm{GL}_n(K)$ be the set of upper triangular matrices, and $U_n \subset B_n$ the subset consisting of those matrices which are 1 on the diagonal.

   (a) Show that B_n and U_n are both closed connected subgroups of $\mathrm{GL}_n(K)$.

   (b) Show that $U_n = B_n'$, and that B_n/U_n is isomorphic to the multiplicative group $(K^\times)^n$.

   (c) Show that the r-th derived subgroup $U_n^{(r)}$ consists of those matrices having zeros on the first r upper diagonals, so in particular $U_n^{(n-1)}$ is trivial. Hence show that $U_n^{(r)} \lhd U_n$ is normal, and for all $0 \leq r < n$ the quotient $U_n^{(r)}/U_n^{(r+1)}$ is isomorphic to the additive group K^{n-r}.

> This shows that B_n and U_n are both solvable groups. In fact, since $U_n^{(r)}$ equals the subgroup generated by all commutators $[u,v]$ with $u \in U_n$ and $v \in U_n^{(r-1)}$, we see that U_n is even a nilpotent group.

3. In this question we consider the derived subgroup G' of a connected algebraic group G.

   (a) Suppose $U, V \subset G$ are dense open subsets. Show that every element $g \in G$ can be written as $g = uv$ for some $u \in U$ and $v \in V$. In other words, $G = UV$.
   Hint. Consider $U \cap gV^{-1}$. Why is this set non-empty?

   (b) Suppose $H \subset G$ is a dense open subgroup. Show that $H = G$.
   Hint. Use the previous part with $U = V = H$.

(c) Consider

$$\phi_n : (G \times G)^n \to G, \quad (g_1, h_1, \ldots, g_n, h_n) \mapsto [g_1, h_1] \cdots [g_n, h_n].$$

Show that we have an increasing sequence of closed irreducible subsets

$$\overline{\mathrm{Im}(\phi_1)} \subset \overline{\mathrm{Im}(\phi_2)} \subset \cdots$$

Explain why this sequence must stabilise, so there exists n such that $\overline{\mathrm{Im}(\phi_r)} = \overline{\mathrm{Im}(\phi_n)}$ for all $r \geq n$.
Deduce that $\overline{\mathrm{Im}(\phi_n)}$ is a closed connected subgroup of G, and that $\mathrm{Im}(\Phi_n) \subset G' \subset \overline{\mathrm{Im}(\phi_n)}$.
Use the previous part to deduce that $G' = \overline{\mathrm{Im}(\phi_n)}$ is a closed connected subgroup of G.

> If $H \leq G$ is a closed subgroup, then the set of cosets G/H is naturally a space with functions, and is in fact a variety. Moreover the natural map $G \to G/H$ is a morphism of varieties.
> In particular, if $N \leq G$ is a closed normal subgroup, then G/N is again an algebraic group, and the natural map $G \to G/N$ is a morphism of algebraic groups.
> Finally, if G is affine and $N \lhd G$ is a closed normal subgroup, then the quotient G/N is again affine.

4. Consider the group $G = U_2$ acting on K^2 via matrix multiplication.
   (a) Describe all the orbits of G on K^2. Hence show that all orbits are closed.
   (b) Show that the fixed point subalgebra $K[X, Y]^G$ is just $K[Y]$.
   (c) Show that the induced topology on the quotient K^2/G is the cofinite topology, so the proper closed sets are just the finite sets.

> It is possible to show that the space with functions K^2/G is not a variety, and hence this is not a geometric quotient.

To be handed in by 29th January.