Non-commutative Algebra 3, SS 2020

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Exercises 2

There were several open questions involving affine spaces, which we now want to address. Let K be an algebraically closed field. We recall the following form of Hilbert's Nullstellensatz.

Nullstellensatz. *Let A be a finitely generated (commutative) K-algebra.*

- *If $\mathfrak{m} \lhd A$ is a maximal ideal, then the natural map $K \to A/\mathfrak{m}$ is an isomorphism.*
- *If $I \lhd A$ is any ideal, then its radical $\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n\}$ is the intersection of all maximal ideals containing I, so $\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$.*

1. Let A be a finitely generated, commutative K-algebra. Assume further that A is reduced, so that $\operatorname{nil} A = 0$. Define $\operatorname{Spec} A$ to be the set of all maximal ideals of A. In this question we will show how to turn this into a space with functions.

   (a) Show first that we can regard A as an algebra of functions on $\operatorname{Spec} A$ via $a(\mathfrak{m}) := a + \mathfrak{m} \in A/\mathfrak{m} \cong K$, where $a \in A$ and $\mathfrak{m} \in \operatorname{Spec} A$. In other words, show that this induces an injective algebra map from A to the algebra of all functions $\operatorname{Spec} A \to K$.

   Note that $a \in \mathfrak{m}$ if and only if $a(\mathfrak{m}) = 0$. In other words, $\mathfrak{m}$ is the kernel of the map $A \to K$, $a \mapsto a(\mathfrak{m})$.

   If we now take $\mathcal{B} := \{\emptyset, \operatorname{Spec} A\}$ and $\mathcal{O}'(\operatorname{Spec} A) := A$, then we can equip $\operatorname{Spec} A$ with the structure of a space with functions as in Exercise Sheet 1, Question 4. Moreover, if X is any space with functions, then a map $\theta\colon X \to \operatorname{Spec} A$ is a morphism of spaces with functions if and only if $a\theta \in \mathcal{O}(X)$ for all $a \in A$.

   (b) Let $\theta\colon X \to \operatorname{Spec} A$ be a morphism of spaces with functions. Show that $a \mapsto a\theta$ is a K-algebra homomorphism $\phi\colon A \to \mathcal{O}(X)$. Moreover, for each $x \in X$, the maximal ideal $\theta(x) \in \operatorname{Spec} A$ is precisely the kernel of the map $A \to K$, $a \mapsto \phi(a)(x)$.

   (c) Conversely, given a K-algebra homomorphism $\phi\colon A \to \mathcal{O}(X)$, show that there is a map $\theta\colon X \to \operatorname{Spec} A$ sending $x \in X$ to the kernel of $A \to K$, $a \mapsto \phi(a)(x)$. Show that $a\theta = \phi(a) \in \mathcal{O}(X)$. Deduce that θ is a morphism of spaces with functions.

   (d) Show that these two constructions are mutually inverse, so yield a bijection between morphisms $X \to \operatorname{Spec} A$ and K-algebra homomorphisms $A \to \mathcal{O}(X)$.

2. Let $\phi\colon A \to B$ be an algebra homomorphism between two finitely generated, commutative and reduced K-algebras.

   (a) Show that if $\mathfrak{n} \lhd B$ is a maximal ideal, then $\phi^{-1}(\mathfrak{n})$ is a maximal ideal of A.

   (b) Show that the morphism of spaces with functions $\theta\colon \operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to ϕ is precisely the map $\theta(\mathfrak{n}) := \phi^{-1}(\mathfrak{n})$ (c.f. the previous exercise, part (c)).

   (c) Recall that if $I \subset A$, then $V(I) = \{\mathfrak{m} : I \subset \mathfrak{m}\} \subset \operatorname{Spec} A$. Show that $\operatorname{Im}(\theta) \subset V(I)$ if and only if $I \subset \operatorname{Ker}(\phi)$. Deduce that $\overline{\operatorname{Im}(\theta)} = V(\operatorname{Ker}(\phi))$.

   (d) We say that θ is dense if $\overline{\operatorname{Im}(\theta)} = \operatorname{Spec} A$. Show that θ is dense if and only if ϕ is injective.

3. Let A be a finitely generated, commutative and reduced K-algebra, and $I \lhd A$ a radical ideal. In this question we want to show that $\operatorname{Spec}(A/I)$ is isomorphic to $V(I)$ with its induced structure as a space with functions.

   (a) Show that A/I is a finitely generated, commutative and reduced K-algebra.

   (b) Consider the canonical algebra homomorphism $\pi\colon A \to A/I$. Show that the corresponding morphism of spaces with functions $\theta\colon \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ with image contained in $V(I)$. Using Exercise Sheet 1, Question 3 we deduce θ induces a morphism of spaces with functions $\theta\colon \operatorname{Spec}(A/I) \to V(I)$.

   (c) To see that θ is an isomorphism, we need to construct its inverse. Show that the restriction map $A \to \mathcal{O}(V(I))$ induces an algebra homomorphism $A/I \to \mathcal{O}(V(I))$, and hence corresponds to a morphism of spaces with functions $\phi\colon V(I) \to \operatorname{Spec}(A/I)$. Show further that $\phi(\mathfrak{m}) = \mathfrak{m}/I$ for all maximal ideals $\mathfrak{m} \supset I$, and hence that ϕ and θ are mutually inverse.

   (d) Give an example of a continuous map $\theta\colon X \to Y$ between two topological spaces such that θ is bijective but not a homeomorphism.

4. Again let A be a finitely generated, commutative and reduced K-algebra. Given $0 \neq a \in A$, we can form the algebra $A_a := A[t]/(1 - at)$. In this question we want to show that $\operatorname{Spec} A_a$ is isomorphic to $D(a) := \{\mathfrak{m} : a \notin \mathfrak{m}\} \subset \operatorname{Spec} A$, with its induced structure as a space with functions.

   (a) Show that A_a is a finitely generated, commutative and reduced K-algebra.

   (b) Consider the canonical algebra map $\pi\colon A \to A_a$. Show that the corresponding morphism of spaces with functions $\theta\colon \operatorname{Spec} A_a \to \operatorname{Spec} A$ with image contained in $D(a)$. Using Exercise Sheet 1, Question 3 we deduce θ induces a morphism of spaces with functions $\theta\colon \operatorname{Spec} A_a \to D(a)$.

   (c) Conversely, show that the restriction map $A \to \mathcal{O}(D(a))$ induces an algebra homomorphism $A_a \to \mathcal{O}(D(a))$, and hence corresponds to a morphism of spaces with functions $\phi\colon D(a) \to \operatorname{Spec} A_a$.

   (d) Show that ϕ and θ are mutually inverse.

5. Let A be a finitely generated, commutative, reduced K-algebra. In this question we want to show that $\mathcal{O}(\operatorname{Spec} A) = A$. In particular, combining this with the previous two exercises yields $\mathcal{O}(D(a)) = A_a$ and $\mathcal{O}(V(I)) = A/I$.

   (a) Suppose $f \in A$ is zero as a function $D(g) \to K$. Show that fg is zero on all of $\operatorname{Spec} A$, and hence that $fg = 0$ as elements of A.

   (b) Take $\theta \in \mathcal{O}(\operatorname{Spec} A)$. Then we can write $\operatorname{Spec} A = \bigcup_\alpha D(g'_\alpha)$ such that $\theta = f'_\alpha/g'_\alpha$ on $D(g'_\alpha)$. Set $f_\alpha := f'_\alpha g'_\alpha$ and $g_\alpha = (g'_\alpha)^2$. Show the following.

      (i) $D(g_\alpha) = D(g'_\alpha)$.

      (ii) $\theta = f_\alpha/g_\alpha$ on $D(g_\alpha)$.

      (iii) $f_\alpha g_\beta = f_\beta g_\alpha$ on all of $\operatorname{Spec} A$, and hence $f_\alpha g_\beta = f_\beta g_\alpha$ as elements of A.

   Hint. For (iii) we know that $f'_\alpha/g'_\alpha = f'_\beta/g'_\beta$ on $D(g'_\alpha g'_\beta)$. Write this in the form $f''/g'_\alpha g'_\beta = 0$ and use the first part.

   (c) Let I be the ideal generated by the g_α. Show that $V(I) = \emptyset$, and hence that $I = A$. Deduce that we can write $1 = \sum_\alpha s_\alpha g_\alpha$ as a finite sum.

   (d) Set $h := \sum_\alpha s_\alpha f_\alpha$, again a finite sum, so $h \in A$. Show that $hg_\beta = f_\beta$, so that $h = f_\beta/g_\beta = \theta$ on $D(g_\beta)$. Deduce that $\theta = h$ as functions $\operatorname{Spec} A \to K$.

To be handed in by 11th May.