Non-commutative Algebra 3, SS 2020

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Exercises 3

A usual, K denotes an algebraically closed field. We define an affine K-algebra to be a finitely generated, commutative and reduced K-algebra.

1. Let X and Y be affine varieties. We want to show that $K[X \times Y] \cong K[X] \otimes_K K[Y]$. This was used in various places in the lecture notes, for example in the sections on Hopf algebras and comodules.

   Let A and B be affine K-algebras, and consider the K-algebra $A \otimes_K B$.

   (a) Let $\mathfrak{m} \lhd A$ be a maximal ideal. Show that $(A \otimes B)/(\mathfrak{m} \otimes B) \cong B$.
   Hint: you will need the Nullstellensatz, as given on the previous exercise sheet.

   (b) Let b_i be a K-basis for B, and let $c = \sum_i a_i \otimes b_i \in A \otimes B$, so that almost all a_i are zero. Show that if c is nilpotent, then each a_i lies in every maximal ideal of A. Deduce that $A \otimes B$ is reduced.

   (c) Prove that $A \otimes_K B$ is again an affine K-algebra.

   (d) Recall that if X is an affine variety, then $K[X]$ is an affine K-algebra, and the assignment $X \mapsto K[X]$ determines a dualty between the category of affine varieties and the category of affine K-algebras.
   Deduce that if X, Y are affine varieties, then $K[X \times Y] \cong K[X] \otimes K[Y]$.
   Hint: a duality swaps producs and coproducts.

2. This question deals with the Segre embedding. Let x_i $(i = 0, \ldots, m)$ be the co-ordinates on $\mathbb{P}^m$, y_p $(p = 0, \ldots, n)$ the co-ordinates on $\mathbb{P}^n$, and z_{ip} $(i = 0, \ldots, m$ and $p = 0, \ldots, n)$ the co-ordinates on $\mathbb{P}^{mn+m+n}$. (It's helpful to write the z_{ip} as a matrix, rather than as a vector.)

   We define a map of sets $\phi \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$, $([x_i], [y_p]) \mapsto ([x_i y_p])$.

   (a) Show that this map is well-defined and injective.

   (b) Let $Z \subset \mathbb{P}^{mn+m+n}$ be the closed subset given by the vanishing of all the polynomials of the form $Z_{ip}Z_{jq} - Z_{iq}Z_{jp}$. Show that ϕ has image Z.
   Hint: for the surjectivity, given a point $z \in Z$ we have $z_{ip} \neq 0$ for some i, p. Set $x_j := z_{jp}/z_{ip}$ and $y_q := z_{iq}/z_{ip}$.

   (c) Show further that ϕ restricts to an isomorphism of spaces with functions
   $$\mathbb{A}^m \times \mathbb{A}^n \cong D'(X_i) \times D'(Y_p) \xrightarrow{\sim} Z \cap D'(Z_{ip}).$$

   (d) Deduce that $\phi \colon \mathbb{P}^m \times \mathbb{P}^n \to Z$ is an isomorphism of spaces with functions.

3. Let $\theta \colon X \to Y$ be a continous map of topological spaces.

   Recall that X is connected provided we cannot write $X = X_1 \sqcup X_2$ as a disjoint union of closed subsets. We say that X is irreducible provided we cannot write $X = X_1 \cup X_2$ as a union of proper closed subsets, so with $X_1, X_2 \neq X$.

(a) Show that X connected implies $\overline{\theta(X)}$ is connected.

(b) Show that X irreducible implies $\overline{\theta(X)}$ is irreducible.

(c) Show that X is irreducible if and only if every non-empty open subset is dense, so $U \subset X$ open non-empty implies $\bar{U} = X$.

4. We have the surjective map of sets $\pi \colon \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$. We saw in the lectures that $\mathbb{P}^n$ has the quotient topology, so $U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}(U)$ is open. It follows that π is continuous.

   Show that π is a morphism of spaces with functions. In other words, show that if $U \subset \mathbb{P}^n$ is open, and $f \in \mathcal{O}(U)$, then $f\pi \in \mathcal{O}(\pi^{-1}(U))$.

5. Here is an example of a space with functions which has an open covering by two affine varieties, but is not separated, so is not a variety.

   The construction is similarly to that of $\mathbb{P}^1$, except we glue the two affine pieces in such a way that we have a 'double origin'.

   Let $X = K \cup \{0'\}$ with the cofinite topology, so every proper closed subset is finite. We set $K' := X - \{0\}$, which we can again identify with $\mathbb{A}^1$.

   For an open subset $U \subset X$ we define $\mathcal{O}(U)$ as follows:

   If $0' \notin U$, then $U \subset K \cong \mathbb{A}^1$ and we can define $\mathcal{O}(U)$.

   If $0 \notin U$, then $U \subset K' \cong \mathbb{A}^1$ and we can define $\mathcal{O}(U)$.

   If $U = X$ then we set $\mathcal{O}(X) = K[t]$, where t is the usual co-ordinate function on K and sends $0'$ to zero.

   (a) Show that this construction determines a space with functions.

   (b) Show that X is not separated.

   Hint: the complement of the diagonal Δ_X in $X \times X$ contains the subset $K \times \{0'\}$. Show that any open subset of $K \times K' \cong \mathbb{A}^2$ containing $K \times \{0'\}$ necessarily contains a point (a, a) with $a \neq 0$, and hence intersects Δ_X. Thus Δ_X is not closed.

To be handed in by 18th May.